SCIENCE

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MSS. intended for publication and books, etc., intended for review should be sent to the responsible editor, Professor J. McKeen Cattell, Garrison-on-Hudson, N. Y.

THE CENTURY'S PROGRESS IN APPLIED MATHEMATICS.

II.

ANOTHER question of widely general, and of peculiar mathematical interest, is the problem first attacked by Fourier, of the distribution and consequent effects of the earth's internal heat. The most interesting phase of this question is that which relates to the time that has elapsed since the crust of the earth became stable and sufficiently cool to support animal life. It is now nearly forty years since Lord Kelvin* startled geologists especially by telling them that Fourier's theory of heat conduction forbids anything like such long intervals of time as they were in the habit of assigning to the aggregate of paleontological phenomena. On several occasions since then Kelvin has restated his arguments with a cogency that has silenced most geologists if it has not convinced most mathematicians. Quite recently, however, the question has become somewhat less one-sided, since geologists and paleontologists are beginning to defend their positions† while that of

* In a memoir 'On the secular cooling of the earth,' *Trans. Royal Society of Edinburgh*, 1862. Republished in Kelvin and Tait's Treatise on Natural Philosophy, appendix D. Kelvin's latest paper on this subject is entitled 'The age of the earth as an abode fitted for life,' and is published in *Philosophical Magazine*, January, 1899; also in SCIENCE, May 12, 1899.

† See Professor T. C. Chamberlin's paper, "Lord

Kelvin is being attacked from the mathematical side.* My own views on this subject were expressed somewhat at length ten years ago, in the address already referred to, and it seems unnecessary here to go into the matter any further than to reaffirm my conviction that the geologists have adduced the weightier arguments. Beautiful as the Fourier analysis is, and absorbingly interesting as its application to the problem of a cooling sphere † is, it does not seem to me to afford anything like so definite a measure of the age of the earth as the visible processes and effects of stratification to which the geologists appeal. In short, the only definite results which Fourier's analysis appears to me to have contributed to knowledge concerning the cooling of our planet are the two following, namely: first, that the process of cooling goes on so slowly that nothing less than a million years is a suitable time unit for measuring the historical succession of thermal events; and secondly, that this process of cooling goes on substantially as if the earth possessed neither oceans nor atmosphere.

It was the well-founded boast of Laplace in the early part of the century that astronomy is the most perfect of the sciences;* and expert contemporary opinion, as we have seen in the case of no less a personage than Green, agreed that the 'Mécanique Céleste' left little room for further advances. Indeed, it would appear that the completeness and the brilliancy of the developments of celestial dynamics during the half century ending with 1825 (the period of Laplace's activity) completely overshadowed all other sciences and retarded to some extent the progress of astronomy itself. The stupendous work of Laplace was chiefly theoretical, however, and he was content in most cases with observational data which accorded with theory to terms of the first order of approximation only. He was probably not so profoundly impressed as men of science at this end of the century are with the necessity of testing a theory by the most searching observational means available. In fact, in elaborating his methods and in applying his theories to the members of the solar system, it has been essential for his disciples to display a degree of ingenuity and a persistence of industry worthy of the master himself. But the prerequisite to progress in celestial mechanics consisted not so much in following up immediately the lines of investigation laid down by Laplace, as in perfecting the methods and in increasing the data of observational astronomy.

Kelvin's address on the age of the earth as an abode fitted for life,' SCIENCE, June 30, 1899; also Sir Archibald Geikie's presidential address to Geological Section of the British Association for the Advancement of Science, Dover meeting, 1899.

* Notably by Professor John Perry. See *Nature*, January 3, and April 18, 1895.

† I have recast this problem of Fourier in two papers published in the *Annals of Mathematics*, Vol. III., pp. 75–88 and pp. 129–144. The solution there given is the only one, so far as I am aware, which lends itself to computation for all values of the time in the history of cooling. A point of much mathematical interest on which this solution depends is the equivalence of the two following series:

$$ru = \frac{2r_0u_0}{\pi}\sum_{n=1}^{n=\infty}\left(\frac{-1}{n}\right)^{n+1} e^{-a^2(n\pi/r_0)^2t}\sin n\pi\frac{r}{r_0},$$

$$= ru_0 - \frac{2r_0u_0}{\sqrt{\pi}}\sum_{n=0}^{n=\infty}\int_{(2n+1)m_0-m}^{(2n+1)m_0+m} e^{-z^2}dz.$$

In these u is the temperature at a distance r from the center of the sphere at any time t; u_0 is the initial uniform temperature of the sphere; r_0 is the radius of the surface of the sphere; a^2 is the diffusivity, supposed constant; and $m = r/(2\sqrt{t})$, $m_0 = r_0/(2a\sqrt{t})$. It will be observed that when the first series (which is Fourier's) converges very slowly, the second converges very rapidly, and vice versa. It will be seen also, that the series refuse, as they should, to give values of the temperature corresponding to negative values of the time.

* "L'Astronomie, par la dignité de son objet et par la perfection de ses théories, est le plus beau monument de l'esprit humain, le titre le plus noble de son intelligence." Système du Monde, Ed., 1884, p. 486.

The development of this branch of science along with the development of the closely related science of geodesy, is a work essentially of the present century, and must be attributed chiefly to the German school of astronomers led by Gauss and Bessel. It is to these eminent minds, as well known in pure as in applied mathematics, that we are indebted for the theories, and for the most advantageous methods of use, of instrumental appliances, and for the refined processes of numerical calculation which secure the best results from observational data. It is a fortunate circumstance, perhaps, considering the irreverence which some modern pure mathematicians show for numerical computations, that Gauss and Bessel began their careers long before the resistless advent of the theory of functions and the theory of groups.

The story of the opportune discovery of the planet Ceres, as related by Gauss himself in the preface to his Theoria Motus Corporum Cœlestium, is well known; but it is less well known that the merit of this magnificent work lies rather in the model groups of formulas presented for the precise numerical solution of intricate problems than in the facility afforded for locating the more obscure members of the solar system. Indeed, the works of Gauss and Bessel are everywhere characterized by a clear recognition of the important distinction between those solutions of problems which are, and those which are not, adapted to numerical calculation. They showed astronomers how to systematize, to expedite, and to verify arithmetical operations in ways which were alone adequate to the accomplishment of the vast undertakings which have since been completed in mathematical geodesy and in sidereal astronomy.

Among the most important contributions of these authors to geodesy and astronomy in particular, and to the precise observational sciences in general, is that branch of the theory of probability called the 'method of least squares.' * No single adjunct has done so much as this to perfect plans of observation, to systematize schemes of reduction, and to give definiteness to computed results. The effect of the general adoption of this method has been somewhat like the effect of the general adoption by scientific men of the metric system; it has furnished common modes of procedure, common measures of precision, and common terminology, thus increasing to an untold extent the availability of the priceless treasures which have been recorded in the century's annals of astronomy and geodesy.

When we pass from the field of observational astronomy to the more restricted but more intricate field of dynamical astronomy, it is apparent that Laplace and his contemporaries quite underestimated the magnitudes of the mathematical tasks they bequeathed to their successors. Laplace, almost unaided, had performed the unparalleled feat of laying down a complete outline of the 'system of the world'; but the labor of filling in the details of that outline, of bringing every member of the solar system into harmony at once with the simple law of gravitation and with the inexorable facts of observation, is a still greater feat which has taxed the combined efforts of the most acute analysts and the most skillful computers of the preceding and present generation.

It is impossible within the limits of a semi-popular address to do more than mention in the most summary way the extraordinary contributions to dynamical astronomy made especially during the present

* Gauss's fundamental paper in this subject is "Theoria combinationis observationum erroribus minimis obnoxiæ," and dates from 1821. Werke, Band IV.

Bessel's numerous contributions to this subject are found in his "Abhandlungen" cited above.

half century, contributions alike formidable by reason of their bulk and by reason of the complexity of their mathematical details. An account of the theory of the perturbative function, or of the theory of the moon, for example, would alone require space little short of a volume.* To mention only the most conspicuous names, there is the pioneer and essentially prerequisite work of the illustrious Gauss and the incomparable Bessel. There is the remarkable work of the brilliant Leverrier (1811–1877), and the not less brilliant Adams (1819–1892),† well known to popular fame by reason of what may be called their mathematical discovery of the planet Neptune. Then came the monumental 'Tables de la Lune' ‡ from the arithmetical laboratory of the indefatigable Hansen; and this marvellous production was quickly followed (1860) by the equally ponderous, and mathematically more important, 'Théorie du Mouvement de la Lune'§ from the pen of the admirably fertile and industrious Delaunay. And finally, there is the still more elaborate work, bringing this great problem of the solar system well-nigh to completeness of solution, which, by common consent, is credited to the two preceding presidents of the American Mathematical Society. || Probably no mathematico-physical undertakings of the century have yielded so many definite, quantitative results to the permanent stock of knowledge as the researches in dynamical astronomy.

* A good account of the progress in dynamical astronomy from 1842 to 1867 is given by Delaunay in 'Rapport sur les Progrès de l'Astronomie,' Paris, 1867.

† The papers of Adams have been edited by Professor W. G. Adams and supplied with a biographical memoir by Professor J. W. L. Glaisher, under the title 'Scientific Papers of John Couch Adams,' Cambridge, at the University Press, Vol. I., 1896.

‡ Published by the British government in 1857.

§ *Mémoires de l'Académie des Sciences de l'Institut Impérial de France*, Tomes XXVIII., XXIX.

|| For an account of the more recent work of Gyldén and Poincaré, reference is made to the presidential address of Dr. G. W. Hill, "Remarks on the progress of celestial mechanics since the middle of the century"; *Bulletin American Mathematical Society*, 2d series, Vol. II., No. 5, p. 125.

But notwithstanding the astonishing degree of perfection to which this science has been brought, there are still some outstanding discrepancies which indicate that the end of investigation is yet a long way off. The moon, which has given astronomers as well as other people, more trouble than any other member of the solar system, is still devious to the extent of a few seconds in a century. The earth, also, it is suspected, is irregular as a time-keeper by a minute but sensible amount;* while it has been proved recently by the exquisite precision of modern observations, that the earth's axis of rotation wanders in a complex way through small but troublesome angles from its mean position, thus causing variations in the astronomical latitude of a place.†

* The effect of tidal friction on the speed of rotation of the earth appears to have been first explained by Ferrel in a 'Note on the influence of the tides in causing an apparent acceleration of the moon's mean position.' This paper was read before the American Academy of Arts and Sciences, in December, 1864, only a few weeks before Delaunay read a similar paper before the French Academy. See Ferrel's autobiography cited above. See also Delaunay's account of his own work in 'Rapport sur les progrès de l' astronomie,' Paris, 1867.

† The cause of such variations is found in the relative mobility of the parts of the earth, especially in the mobility of the oceans and atmosphere. Three types of variation may occur, namely: 1st, that due to sudden changes in the relative positions of the parts of the earth's mass; 2d, that due to secular changes in position of those parts; and 3d, that due to periodic shiftings of those parts. Of these the most important appears to be the periodic type. A surprising, and as yet not fully explained, discrepancy brought to light by the discovery of latitude variations is the fact that the instantaneous axis of rotation of the earth makes a complete circuit around the axis of figure in about 428 days, instead of in about 305 days as has been supposed from the time of Euler down to the present decade. The discovery of this discrepancy is due to

A question of intense interest to astronomers in the early part of the century is that of the stability of the solar system. Lagrange, Laplace, and Poisson thought they had demonstrated that, whatever may have been the origin of this system, the existing order of events will go on indefinitely. This conclusion seems to have been alike satisfactory to scientific and unscientific men. But with the growth of the doctrine of energy and with the developments of thermodynamics, it has come to appear highly probable that the solar system has not only gone through a long series of changes in the past, but is destined to undergo a similarly long series of vicissitudes in the future. In other words, while our predecessors of a century ago thought the 'system of the world' stable, our contemporaries are forced to consider it unstable.*

But interesting as this question of stability still is, there is no pressing necessity, fortunately, for a determination of the ulterior fate of our planet. A more important question lies close at hand, and merits, it seems to me, immediate and serious investigation. This question is the fundamental one whether the beautifully simple law of Newtonian attraction is exact or only approximate. No one familiar with celestial mechanics or with the evidence for the law of gravitation as marshalled by Laplace in his 'Système du Monde' can fail to appreciate the reasons for the profound conviction, long held by astronomers, that this law is exact. But on the other hand no one acquainted with the obstinate properties of matter can now be satisfied with the Newtonian law until it is proved to hold true to a much higher degree of approximation than has been attained hitherto.* For, in spite of the superb experimental investigations made particularly during the past quarter of a century by Cornu and Baille,† Poynting,‡ Boys,§ Richarz and Krigar-Menzel,|| and Braun,¶ it must be said that the gravitation constant is uncertain by some units in the fourth significant figure, and possibly by one or two units even in the third figure;** thus falling, along with the sun's parallax, the annual stellar aberration, and the moon's mass, amongst the least well determined constants of the solar system. Here then is a fruitful field for research. The direct measurement of the gravitation constant to a much higher degree of precision seems to

Dr. S. C. Chandler and was announced in the *Astronomical Journal*, No. 248, November, 1891. For the mathematical theory of this subject and for titles of the principal publications bearing on this theory, reference may be made to the author's paper on 'Mechanical interpretation of variations of latitudes,' *Astronomical Journal*, No. 345, May, 1895; and to a paper by S. S. Hough on 'The rotation of an elastic spheroid,' *Philosophical Transactions*, No. 187, 1896.

* See a review of this subject by M. H. Poincaré, "Sur la stabilité du système solaire," in *Annuaire du Bureau des Longitudes*, for 1898.

* As to the degree of precision with which the Newtonian law is established by astronomical data, see Professor Newcomb's "Elements of the four inner planets and the fundamental constants of astronomy," Supplement to American Ephemeris and Nautical Almanac for 1897, Washington, 1895.

† *Comptes rendus*, LXXVI., 1873.

‡ The Mean Density of the Earth, by J. H. Poynting, Chas. Griffin & Co., London, 1894.

§ *Philosophical Transactions*, No. 186, 1895.

|| Sitzungsberichte, Berlin Academy, Band 2, 1896.

¶ *Denkschriften*, Math. Natur. Classe, Vienna Academy, Band LXIV., 1897.

** The results of the investigators mentioned for the gravitation constant are, in C. G. S. units, as follows, the first result having been computed from data given by MM. Cornu and Baille in the publication referred to:

Cornu and Baille (1873)	6668×10^{-11}
Poynting (1894)	6698×10^{-11}
Boys (1894)	6657×10^{-11}
Richarz and Krigar-Menzel (1896)	6685×10^{-11}
Braun (1897)	6658×10^{-11}

Regarding these as of equal weight, their mean is 6673×10^{-11} with a probable error of ± 5 units in the fourth place, or 1/1330th part. This is of about the same order of precision as that deduced by Professor Newcomb from astronomical data.

present insuperable obstacles; but may not the result be reached by indirect means, or may it not be possible to make the solar system break its Sphinx-like reticence of the centuries and disclose the gravitational mechanism itself?

Just as the theories of astronomy and geodesy originated in the needs of the surveyor and navigator, so has the theory of elasticity grown out of the needs of the architect and engineer. From such prosaic questions, in fact, as those relating to the stiffness and the strength of beams, has been developed one of the most comprehensive and most delightfully intricate of the mathematico-physical sciences. Although founded by Galileo, Hooke, and Mariotte in the seventeenth century, and cultivated by the Bernoullis and Euler in the last century, it is, in its generality, a peculiar product of the present century.* It may be said to be the engineer's contribution of the century to the domain of mathematical physics, since many of its most conspicuous devotees, like Navier, Lamé, Rankine, and Saint-Venant, were distinguished members of the profession of engineering; and it is a singular circumstance that the first of the great originators in this field, Navier, should have been the teacher of the greatest of them all, Barré de Saint-Venant.† Other great names are also prominently identified with the growth of this theory and with the recondite problems to which it has given rise. The acute analysts, Poisson, Cauchy, and Boussinesq, of the French school of elasticians; the profound physicists, Green, Kelvin, Stokes, and Maxwell, of the British school; and the distinguished Neumann (Franz Ernst, 1798–1895), Kirchhoff (1824-1887), and Clebsch (1833–1872), of the German school; have all contributed heavily to the aggregate of concepts, terminology, and mathematical machinery which make this at once the most difficult and the most important of the sciences dealing with matter and motion.

The theory of elasticity has for its object the discovery of the laws which govern the elastic and plastic deformation of bodies or media. In the attainment of this object it is essential to pass from the finite and grossly sensible parts of media to the infinitesimal and faintly sensible parts. Thus the theory is sometimes called molecular mechanics, since its range extends to infinitely small particles of matter if not to the ultimate molecules themselves. It is easy, therefore, considering the complexity of matter as we know it in the more elementary sciences, to understand why the theory of elasticity should present difficulties of a formidable character and require a treatment and a nomenclature peculiarly its own.

While it would be quite inappropriate on such an occasion to go into the mathematical details of this subject, I would recall your attention for a moment to the surprisingly simple and beautiful concepts from which the mathematical investigation proceeds rapidly and unerringly to the equations of equilibrium or motion of a particle of any medium. The most important of these are the concept which relates to the stresses on the particle arising from its connection with adjacent parts of the medium, and

* An admirable history of this science, dealing with its technical aspects, was projected by Professor Isaac Todhunter and completed by Professor Karl Pearson, under the title "A History of the Theory of Elasticity and the Strength of Materials from the time of Galilei to the present time." Cambridge, at the University Press: Vol. I., Galilei to Saint-Venant, 1886; ol. VII., Parts I. and II., Saint-Venant to Lord Kelvin, 1893.

A capital though abridged history of the science is given by Saint-Venant in his annotated edition of Navier's Résistance des Corps Solides, troisième édition, Paris, 1864.

The history of Todhunter and Pearson is dedicated to Saint-Venant, who has been fitly called 'the dean of elasticians.'

† And this illustrious master has left a worthy pupil in M. J. Boussinesq, Professor in the Faculty of Sciences, Paris.

the concept with regard to the distortions of the particle which result from the stresses. If the particle be a rectangular parallelopiped, for example, these stresses may be represented by three pressures or tensions acting perpendicularly to its faces together with three tensions acting along, or tangentially to, those faces. Thus the adjacent parts of the medium alone contribute six independent force components to the equations of equilibrium or motion; and the stresses, or the amounts of force per unit area, which produce these components are technically known as tractions or shears according as they act perpendicularly to or tangentially along the sides of the particle.* Corresponding to these six components there are six different ways in which the particle may undergo distortion. That is, it may be stretched or squeezed in the three directions parallel to its edges; or, its parallel faces may slide in three ways relatively to one another. These six different distortions, which lead in general to a change in the relative positions of the ends of a diagonal of the parallelopiped, are measured by their rates of change, technically called strains, and distinguished as stretches or slides according as they refer to linear or angular distortion.†

It is from such elementary dynamical and kinematical considerations as these that this theory has grown to be not only an indispensable aid to the engineer and physicist, but one of the most attractive fields for the pure mathematician. As Pearson has remarked, "There is scarcely a branch of physical investigation, from the planning of a gigantic bridge to the most delicate fringes of color exhibited by a crystal, wherein it does not play its part."* It is, indeed, fundamental in its relations

* The terminology here used is that of Todhunter and Pearson, History of the Theory of Elasticity and Strength of Materials, Vol. I., Note B.

† The terminology and symbology of the theory of elasticity appear to be more highly developed than those of any other mathematical science. A comparison of the terms and symbols of elasticity with those of the older subject of hydromechanics, as shown, in part, below, is instructive:

IN ELASTICITY.

Stresses.		Strains.	
Tractions	p_{xx}, p_{yy}, p_{zz}	Stretches	s_x, s_y, s_z
Shears	p_{yz}, p_{zx}, p_{xy}	Slides	σ_{yz}, σ_{zx}, σ_{xy}

Shifts, or components of displacement $\begin{cases} u \\ v \\ w \end{cases}$

Shift-fluxions, or space rates of change of shifts $\begin{cases} \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}, \\ \dfrac{\partial v}{\partial x}, \text{etc.}, \\ \dfrac{\partial w}{\partial x}, \text{etc.}, \end{cases}$

Dilitation, $\theta = s_x + s_y + s_z = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$

Twists $\begin{cases} \tau_{yz} = \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) \\ \tau_{zx} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right) \\ \tau_{xy} = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) \end{cases}$

Displacement potential in irrotational, or pure, strain.

IN HYDROMECHANICS.

Fluid pressure p

Component velocities $\begin{cases} u \\ v \\ w \end{cases}$

Space rates of change of component velocities $\begin{cases} \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}, \\ \dfrac{\partial v}{\partial x}, \text{etc.}, \\ \dfrac{\partial w}{\partial x}, \text{etc.}, \end{cases}$

Expansion, $\theta = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$.

Component spins, or components of molecular rotation $\begin{cases} \xi = \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) \\ \eta = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right) \\ \zeta = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) \end{cases}$

Velocity potential in irrotational motion.

* History of Elasticity, etc., Vol. I., p. 872.

to the theory of structures, to the theory of hydromechanics, to the elastic solid theory of light, and to the theory of crystalline media.

In closing these very inadequate allusions to this intensely practical and abstrusely mathematical science, it is fitting that attention should be called to the magnificent labors of the 'dean of elasticians,' M. Barré de Saint-Venant. It was the object of his life-work to make the theory of elasticity serve the utilitarian purposes of the engineer and at the same time to divest it so far as possible of all empiricism. His epoch-making memoir* of 1853, on the torsion of prisms, is not only a classical treatise from the practical point of view, but one of equal interest and importance in its theoretical aspects. His investigations are everywhere delightfully interesting and instructive to the physicist; and many parts of them are replete, as observed by Kelvin and Tait,† with "astonishing theorems of pure mathematics, such as rarely fall to the lot of those mathematicians who confine themselves to pure analysis or geometry, instead of allowing themselves to be led into the rich and beautiful fields of mathematical truth which lie in the way of physical research." More important still in a didactic sense are his annotated edition of Navier's 'Résistance des Corps Solides,' of 1864, and his annotated edition of the French translation of the 'Theorie der Elasticität fester Körper,' of Clebsch, which appeared in 1883. These monumental works, whose annotations and explanatory notes quite overshadow the text of the original authors, must remain for a long time standard sources of information as to the history, theory, methods and results of this complex subject. The luminous exposition, the penetrating insight into physical relations, and the scientific candor in his criticism of other authors, render the work of Saint-Venant worthy of the highest admiration.

* 'Mémoire sur la torsion des prismes,' etc., published in *Mémoires des savants étrangeres*, Tome XIV., 1855.

† Natural Philosophy, 2d ed., Part II., p. 249.

Closely allied to the theory of elasticity, though historically much older, is the science of hydromechanics. The latter is, indeed, included essentially in the former; and probably the great treatises of the next century will merge them under the title of molecular mechanics. It may seem somewhat singular that the mathematical theory of solids should have originated so many centuries later than the theory of fluids; for at first thought, tangible though flexible solids would appear much more susceptible of investigation than mobile liquids and invisible gases. But a little reflection leads one to the conclusion that it was, in fact, much easier to observe the data essential to found a theory of hydromechanics than it was to discover the principles which led to the theory of stress and strain; and the time interval between Archimedes and Galileò may serve perhaps as a rough measure of the relative complexity of hydrostatics and the theory of flexure and torsion of beams. It must not be inferred, however, that the simplicity of the phenomena of fluids in a state of relative rest extends to the phenomena of fluids in a state of relative motion; for the gap that separates hydrostatics from hydrokinetics is one which has not yet been fully bridged even by the aid of the powerful resources of modern mathematics.

The elements of hydrokinetics, with which branch of hydromechanics this sketch is alone concerned, were laid down by Euler about the middle of the last century.* It

* 'Principes géneraux du mouvement desflui des.' *Histoire de l'Académie de Berlin*, 1755.

'De Principiis motus fluidorum.' *Novi Commentarii Academiæ, Scientiarum Imperialis Petropolitanæ*, Tomus XLV., Pars I., pro anno 1759.

is to him that we owe the equations of motion of a particle of a 'perfect fluid.' This is an ideal fluid, moving without friction, or subject, in technical terminology, to no tangential stress. But while no such fluids exist, it is easily seen that under certain circumstances this assumed condition approaches very closely to the actual condition; and, in accordance with the method of mathematico-physical science in untangling the intricate processes of nature, progress has proceeded by successive steps from the theory of ideal fluids toward a theory of real fluids.

The history of the developments of hydromechanics during this century has been very carefully and completely detailed in the reports to the British Association for the Advancement of Science of Sir George Gabriel Stokes,* in 1846, and of Professor W. M. Hicks,† in 1881 and 1882. And the history of the subject has been brought down to the present time by the address of Professor Hicks before Section A of the British Association for the Advancement of Science in 1895, and by the report‡ of Professor E. W. Brown to Section A of the American Association for the Advancement of Science in 1898. It may suffice here, therefore, to glance rapidly at the salient points which mark the advances from the state of the science as it was left by Lagrange a hundred years ago.

The general problem of the kinetics of a particle of a 'perfect fluid' is easily stated. It is this:* given for any time and for any position of the particle its internal pressure, its density, and its three component velocities, along with the forces to which it is subject from external causes; to find the pressure, density, and velocity components corresponding to any other time and to any other position. There are thus, in general, five unknown quantities requiring as many equations for their determination. The usual six equations of motion, or the equations of d'Alembert, contribute only three to this required number, namely, the three equations of translation of the particle, since the three which specify rotation vanish by reason of the absence of tangential stress. A fourth equation comes from the principle of the conservation of mass, which is expressed by equating the time rate of change of the mass of the particle to zero. This gives what is technically called the equation of continuity. A fifth equation is usually found in the law of compressibility of the fluid considered.†

Now, the equations of rotation, as just stated, refuse to answer the question whether the particles proceed in their

* 'Report on recent researches in hydrodynamics,' Report of British Association for the Advancement of Science for 1846.

† 'Report on recent progress in hydrodynamics,' Reports of British Association for the Advancement of Science for 1881 and 1882.

‡ 'On recent progress towards the solution of problems in hydrodynamics,' Proceedings of American Association for the Advancement of Science for 1898. See also SCIENCE, November 11, 1898.

Reference should be made also to Professor A. E. H. Love's paper 'On recent English researches in vortex-motion,' in the *Mathematische Annalen*, Band XXX., 1887.

* The statement here given is that of the 'historical method,' which seeks to follow a particle of fluid from some initial position to any subsequent position and to specify its changes of pressure, density and speed. What is known as the 'statistical method,' on the other hand, directs attention to some fixed volume in the fluid and specifies what takes place in that volume as time goes on.

† The five equations in question are

$$\frac{du}{dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x},\qquad \frac{d(V\rho)}{dt}=0,$$
$$\frac{dv}{dt}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y},$$
$$\frac{dw}{dt}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z},\qquad p=f(\rho);$$

in which p is the pressure and ρ is the density at the centroid (x, y, z) of the particles; V is its volume; u, v, w are its component velocities; and X, Y, Z are the force components per unit mass arising from external causes.

orbits without rotation or whether they undergo rotation along with their motion of translation. This was a critical question, for the failure to press and to answer it seems to have retarded progress for nearly half a century. Lagrange, and after him Cauchy and Poisson, knew that under certain conditions* the differential equations of motion are integrable, but they do not appear to have understood the physical meaning of these conditions. It remained for Sir George Gabriel Stokes to show that the Lagrangian conditions of integrability correspond to the case of no molecular rotation, thus clearly distinguishing the two characteristic types of what we now call irrotational and rotational motion.† Such was the great step made by Stokes in 1845; and it furnishes a forcible illustration of the importance, in applied mathematics, of attending to the physical meaning of every symbol and every combination of symbols.

Thirteen years later came the remarkable memoir of Helmholtz (1821–1894) on the integrals of the equations of hydrokinetics for the case of rotational, or vortex, motion.‡ This memoir is alike wonderful for the directness with which the mathematical argument proceeds to its conclusions and for the clearness of insight it affords of the physical phenomena discussed. In short, it opened up a new realm and supplied the results, concepts, and methods which led the way to the important advances in the science made during the past three decades.

Another powerful impulse was given to hydrokinetics, and to all other branches of mathematical physics as well, by Kelvin and Tait's Natural Philosophy—the Principia of the nineteenth century—the first edition of which appeared in 1867. From this great work have sprung most of the ideas and methods appertaining to the theory of motion of solids in fluids, a theory which has yielded many interesting and surprising results through the researches of Kirchhoff, Clebsch, Bjerknes, Greenhill, Lamb and others. Of prime importance also are the numerous contributions of Lord Kelvin to other branches of hydrokinetics, and particularly to the theory of rotational motion.* In fact, every department of the entire science of hydromechanics, from the preliminary concepts up to his vortex atom theory of matter, has been illuminated and extended by his unrivalled fertility.

When we turn to the more intricate branch of the subject which deals with the motion of viscous fluids, or with the motion of solids in such fluids, it appears that the progress of the century is less marked, but still very noteworthy. This branch is closely related to the theory of elasticity, and goes back naturally to the early researches of Navier, Poisson and Saint-Venant; but the revival of interest in this, as well as in the less intricate branch of the subject, seems to date from the fruitful memoir† of Stokes, of 1845, and from his report to the British Association for the Advancement of Science of 1846. Since then many interesting and useful problems relative to the flow of viscous fluids and to

* That is, when $udx + vdy + wdz$ is a perfect differential, u, v, w being velocity components; or, when

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x},$$

which are the doubles of the components of molecular rotation, vanish, these latter being the conditions for the existence of a velocity potential.

† This discovery of Stokes was announced in his fundamental paper 'On the theories of internal friction of fluids in motion, and of the equilibrium and motion of elastic solids,' *Transactions of the Cambridge Philosophical Society*, Vol. VIII. Reprinted also in his Mathematical and Physical Papers, Vol. I.

‡ "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," *Crelle's Journal für die reine und angewandte Mathematik*, 1858.

* 'On vortex motion,' 1867. *Transactions of the Royal Society of Edinburgh*, Vol. XXV.

† Cited above.

the motion of solids in such media, have been successfully worked out to results which agree fairly well with experiment. But on the whole, notwithstanding the searching investigations in this field of Stokes, Maxwell, Helmholtz, Boussinesq, Meyer, Oberbeck, and many others, it must be said that difficulties, both in theory and in experiment, of a formidable character remain to be surmounted.*

Of all branches of hydromechanics there is none of so great practical utility and of such widely popular interest as the theory of tides and waves. These phenomena of the sea are appreciable to the most casual observer; and there has been no lack of impressive descriptions of their effects from the days of Curtius Rufus down to the present time. The mechanical theory of tides and waves is, however, a distinctly modern development whose perfection must be credited to the labors of the mathematicians of the present century.†

Here, again, progress is measured from the advanced position occupied by Laplace, who was the first to attempt a solution of the tidal problem on hydrokinetic principles. After the fundamental contributions of Laplace, contained in the second and fifth volumes of the 'Mécanique Céleste,' the next decisive advance was that made by Sir George Airy (1801–1892), in his article on tides and waves, which appeared in the Encyclopædia Metropolitana in 1842. A quarter of a century later came the renaissance, started undoubtedly by the great memoir of Helmholtz and by the Natural Philosophy of Kelvin and Tait, along with Lord Kelvin's inspiring communications on almost every phase of wave and tidal problems to scientific societies and journals. Then followed the decided theoretical improvements in tidal theory of Professor William Ferrel,* particularly in the development of the tide generating potential and in the determination of the effects of friction. And a little later there appeared the novel investigations of Professor G. H. Darwin, who, in addition to furnishing a complete practical treatment of terrestrial tides,† has extended tidal theory to the solar system and thrown an instructive light on the evolutionary processes whence the planets and their satellites have emerged and through which they are destined to pass in the future.‡

As we reflect on the progress which has been thus summarily, and quite inadequately outlined, it will appear that the mathematicians of the nineteenth century have contributed a splendid aggregate of permanent accessions to knowledge in the domain of the more exact of the physical sciences. And as we turn from the certain past to the less certain future, one is prone to conjecture whether this brilliant progress is to continue, and, if so, what part the

*An extremely interesting method of experimental investigation has been recently applied with success by Professor Hele-Shaw. See a paper by him on 'Stream-line motion of a viscous film,' and an accompanying paper by Sir G. G. Stokes on 'Mathematical proof of the identity of the stream-lines obtained by means of a viscous film with those of a perfect fluid moving in two dimensions.' Report of British Association for the Advancement of Science for 1898.

† An excellent summary of the history and theory of tides, and of methods of observing and predicting them, is given by Dr. Rollin A. Harris in his 'Manual of Tides,' published as Appendices 8 and 9 of the Report of the U. S. Coast and Geodetic Survey for 1897.

* 'Tidal Researches.' Appendix to Report of U. S. Coast and Geodetic Survey for 1874, Washington, 1874.

† In article on tides in Encyclopædia Britannica, 9th edition.

‡ Darwin's investigations are published in a series of papers in the *Philosophical Transactions of the Royal Society of London*, Parts I., II., 1879; Part II., 1880; Part II., 1881; Part I., 1882. They are republished in part in Appendix G, Thomson and Tait's Natural Philosophy, 2d edition. See also the capital semi-poplar work, 'The Tides and Kindred Phenomena in the Solar System,' by G. H. Darwin Boston and New York, Houghton, Mifflin & Co., 1889.

American Mathematical Society may play in promoting further advances. With respect to these enquiries I should be loath to hazard a prediction or to offer advice. But there appears to be no reason for entertaining other than optimistic expectations. The routes along which exploration may proceed are numerous and attractive. We have only to follow the example set by Laplace, Poisson, Green, Gauss, Maxwell, Kirchhoff, Saint-Venant, Helmholtz, and their eminent contemporaries and successors. In commending the works of these great masters to the younger members especially of the American Mathematical Society, I would not be understood as urging the cultivation of pure mathematics less, but rather as suggesting the pursuit of applied mathematics more. The same sort of fidelity to research and the same sort of genius for infinite industry which enabled those masters to accomplish the grand results of the nineteenth century, may be confidently expected to achieve equally grand results in the twentieth century.

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CRUISE OF THE ALBATROSS.

II.

THE following letter from Dr. Agassiz, dated Papeete Harbor, Tahiti Island, November 6, 1899, has been received by the United States Fish Commission and is here published by courtesy of Commissioner Bowers.

During our stay in Papeete some time was spent in examining that part of the barrier reef of Tahiti which had been surveyed by the *Challenger*. We found the condition of the outer slope of the reef quite different from its description as given in the *Challenger* narrative. The growing corals were comparatively few in number, and the outer slope showed nothing but a mass of dead corals and dead coral bowlders beyond 16 or 17 fathoms, few living corals being observed beyond 10 to 12 fathoms.

We also made an expedition to Point Venus, to determine, if possible, the rate of growth of the corals on Dolphin Bank from the marks which had been placed on Point Venus by Wilkes, in 1839, and by MM. Le Clerk and de Bénazé, of the French navy, in 1869. We found the stones and marks as described, but, in view of the nature and condition of Dolphin Bank, did not think it worth while to make a careful survey, as Captain Moser had intended to do. On examining Dolphin Bank in the steam launch I was greatly surprised to find that there were but few corals growing on it. I could see nothing but sparsely scattered heads, none larger than my fist, the top of the bank being entirely covered by nullipores. We sounded across the bank in all possible directions and examined it thoroughly, and at the stage of water at which we sounded, found about 18 inches difference from the soundings indicated by the charts. It is also greatly to be regretted that Dolphin Bank was not examined, neither in 1839 nor in 1869, and notes made of what species of corals, if any, were growing on its surface; for an excellent opportunity has been lost to determine the growth of corals during a period of 60 years. The choice of this bank as a standard to determine the growth of corals was unfortunate, as it is in the midst of an area comparatively free from corals.

Extensive collections have been made at Papeete during our visit by the naturalists of the *Albatross*.

After refitting and coaling here, we left on the 5th of October for a cruise in the Paumotus.

We steamed for Makatea, which we had visited on our way to Tahiti, and not only examined the island more in detail, but took a number of photographs of the cliffs