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*THE CENTURY'S PROGRESS IN APPLIED MATHEMATICS.**

I.

THE honor of election to the presidency of the American Mathematical Society carries with it the difficult duty of preparing an address, which may be at once interesting and instructive to a majority of the membership, and which may indicate at the same time the lines along which progress may be expected in one or more branches of our favorite science. In partial recognition of the honor you have conferred upon me it has seemed that I could do no better than to consider with you some of the principal advances that have been made in mathematical science during the past century. But here at the outset one must needs feel sharply restricted by the limitations of his knowledge and by the wide extent of the domain to be surveyed. Especially must this be the case with one who belongs to no school of mathematicians, unless it be the 'old school' of inadequate opportunities and desultory training. On account of these conditions, I have found it essential to accept the ordinary division of the science into pure and applied mathematics and to confine my attention in this address wholly to applied mathematics. Here again, however, it is

* Address of the President of the American Mathematical Society, read December 28, 1899.

necessary to impose restrictions, for the domain thus divided is still far too large to be reviewed adequately in the brief interval allotted to the present occasion. I have therefore limited my considerations chiefly to those branches of applied mathematics which were already recognized as such at the beginning of the present century. The most important of these branches appear to be analytical mechanics, geodesy, dynamical astronomy, spherical or observational astronomy, the theory of elasticity, and hydromechanics. This rather arbitrary subdivision may be made to include several important branches not enumerated, while it must exclude others of equal or greater importance. Thus the theory of heat diffusion which led Fourier to the wonderful analysis which bears his name may be alluded to under physical geodesy; the theories of sound and light may be regarded as applications merely of the theories of elasticity and hydromechanics; while the theories of electricity, magnetism, and thermodynamics, which are the peculiar and perhaps most important developments of the present century, must be excluded almost altogether.

Another difficulty which besets one who would speak of the progress in question is that arising from the technicalities of the subjects to be discussed. Beautiful and important as these subjects are when arrayed in their mathematical dress, and thrilling as they truly are when rehearsed with appropriate terminology in the quiet of one's study, it must be confessed that they are on the whole rather uninviting for the purposes of semi-popular exposition. In order to meet this difficulty it seems best to relegate technicalities which demand expression in symbols to footnotes and, while freely using technical terminology, to translate it into the vernacular whenever essential. Thus it is hoped to avoid the dullness of undue condensation on the one hand, and the superficiality of mere literary description on the other.

The end of the last century marks one of the most important epochs in the history of mathematical science. This time, one hundred years ago, the master work of Lagrange (1736–1813), the Mécanique Analytique, had been published about eleven years. The first two volumes of the Mécanique Céleste of Laplace (1749–1827), undoubtedly the greatest systematic treatise ever published, had just been issued. Fourier (1768–1830), whose mathematical theory of heat was destined to play a wonderful rôle in pure and applied mathematics, was a soldier statesman in Egypt, where with Napoleon he stood before the pyramids while the centuries looked down upon them.* Gauss (1777–1855), who with Lagrange and Cauchy (1789–1857) must be ranked among the founders of modern pure mathematics, was a promising but little known student whose Disquisitiones Arithmeticae and other papers were soon to win him the directorship of the observatory at Göttingen. Poisson (1781–1840), to whom we owe in large part the beginnings of mathematical physics, had just started on his brilliant career as a student and professor in the École Polytechnique. Bessel (1784–1846), whose theories of observational astronomy and geodesy were destined soon to assume a prominence which they still hold, was an accountant in a trading house at Bremen. Dynamical astronomy, the favorite science of the day was under the dominating genius of Laplace, with no one to dispute his preëminence, and with only Lagrange and Poisson as friendly competitors in the same field. Rational mechanics as we now know it, was soon to

* The bombastic words of Bonaparte, "Songez que du haut de ces pyramides quarante siècles vous contemplent," may be excused, perhaps, in view of the fact that Fourier, Monge, and Berthollet were present on the occasion.

be simplified and systematized by Poinsot (1777–1859), Poisson, Möbius (1790–1868), and Coriolis (1792–1843), who were all at this time under twenty-five years of age. The undulatory theory of light, in which Young (1773–1829), Fresnel (1788–1827), Arago (1786–1853), and Green (1793–1841) were to be the most conspicuous early figures, was just beginning to be considered as an alternative to the emission theory of Newton. The theory of elasticity, or the theory of stress and strain as it is now called, was about to be reduced to the definiteness of formulas at the hands of Navier (1785–1836), Poisson, Cauchy, and Lamé (1795–1870). Planetary and sidereal astronomy, to which so much of talent, time, and treasure have since been devoted, was soon to receive the fruitful impetus imparted to it by the German school of Gauss, Bessel, Encke (1791–1865), and Hansen (1795–1874).

The advances that have been made during the past century in analytical mechanics must be measured from the elevated standard attained by Lagrange in his Mécanique Analytique. To work any improvement over this, to simplify its demonstrations, or to elaborate its details, was a task fit only for the keenest intellects. Lagrange had, as he supposed, reduced mechanics to pure mathematics. Geometrical reasonings and diagrammatic illustrations were triumphantly banished from this science and replaced by the systematic and unerring processes of algebra. "Ceux qui aiment l'Analyse," he says, "verront avec plaisir la Mécanique en devinir une nouvelle branche, et me sauront gré d'en avoir étendu ainsi le domaine." The mathematical world has not only accepted Lagrange's estimate of his work, but has gone further, and considers his achievement one of the most brilliant and important in the whole range of mathematical science. "The mechanics of Lagrange," as Mach has well said, "is a stupendous contribution to the economy of thought."*

Nevertheless, improvements were essential, and they came in due time. As we can now see without much difficulty, Lagrange and most of his contemporaries in their eagerness to put mechanics on a sound analytical basis overlooked to a serious extent its more important physical basis. The prevailing mathematical opinion was that a science is finished as soon as it is expressed in equations. One of the first to protest against this view was Poinsot, though the preëminent importance of the physical aspect of mechanics did not come to be adequately appreciated until the latter half of the present century. The animating idea of Poinsot was that in the study of mechanics one should be able to form a clear mental picture of the phenomena considered; and that it does not suffice to put the data and the hypotheses into the hopper of our mathematical mill and then to trust blindly to its perfection in grinding the grist. In elaborating this idea he produced two of the most important elementary treatises on mechanics of the century. These are his Éléments de Statique published in 1804, and his Théorie Nouvelle de la Rotation des Corps published in 1834.† In the former work he developed the beautiful and fruitful theory of couples and their composition, and the conditions of equilibrium, as they are now commonly expressed in elementary books.

* The Science of Mechanics, by Dr. Ernst Mach. Translated from the German by Thomas J. McCormack. Chicago, Open Court Publishing Co., 1893.

† Outlined to the Paris Academy in 1834. In the introduction to the edition of 1852 he says, "Voici une des questions qui m'ont le plus souvent occupé, et, si l'on me permet de parler ainsi, une des choses que j'ai le plus désiré de savoir en dynamique.

"Tout le monde se fait une idée claire du mouvement d'un point, * * * Mais, s'il s'agit du mouvement d'un corps de grandeur sensible et de figure quel-conque, il faut convenir qu'on ne s'en fait qu'une idée très-obscure."

In the latter he took up the more recondite question of rendering a clear account of the motion of a rigid body. This problem had been treated already by the illustrious Euler, d'Alembert, Lagrange, and Laplace, and it seemed little short of temerity to hope for any improvement. But Poinsot entertained that hope and his efforts proved surprisingly successful. His little volume of about one hundred and fifty pages is still one of the finest models of mathematical and mechanical exposition; and his repeated warning, "gardons-nous de croire qu'une science soit faite quand on l'a réduite à des formules analytiques," has been fully justified. He gave us what may be called the descriptive geometry of the kinetics of a rotating rigid body, the 'image sensible de cette rotation'; he clarified the theory of moments of inertia and principal axes; he made plain the meaning of what we now call the conservation of energy and the conservation of moment of momentum of systems which are started off impulsively; and he surpassed Laplace himself in expounding the theory of the invariable plane.

Another elementary work of prime importance in the progress of mechanics was Poisson's Traité de Mécanique. Poisson belonged to the Lagrangian school of analysts, but he was so profoundly devoted to mathematical physics that almost all his mathematical work was suggested by and directed towards practical applications. His facility and lucidity in exposition rendered all his works easy and attractive reading, and his treatise on mechanics is still one of the most instructive books on that subject. He was one of the first to call attention to the value of the principle of homogeneity in mechanics,* a principle which, as expanded in Fourier's theory of dimensions,† has proved of the greatest utility in the latter half of the century. The influence of Poisson's work in mechanics proper, very widely extended, of course, by his memoirs in all departments of mathematical physics, is seen along nearly every line of progress since the beginning of the century.

Of other works which paved the way to the present advanced state of mechanical science, it may suffice to mention the Cours de Mécanique* of Poncelet (1788–1867), the Traité de Mécanique des Corps Solides et de l'Effet des Machines† of Coriolis, and the Lehrbuch der Statik‡ of Möbius. To the two former of these authors we owe the fixation of ideas and terminology concerning the doctrine of mechanical work, while the suggestive treatise of Möbius foreshadowed a new type of mechanical concepts since cultivated by Hamilton (Sir W. R., 1805–1865). Grassmann (1809–1877), and others under the general designation of vector analysis.

Following close after the development of the elementary ideas whose history we have sketched came the important improvements in the Lagrangian analysis due to Hamilton.§ With these additions of Hamilton, amplified and clarified by the labors of Jacobi, Poisson, and others,‖ analytical mechanics may be said to have reached its present degree of perfection so far as mathematical methods are concerned. By these methods every mechanical question may be stated in either of three characteristic though interconvertible ways, namely: by the equations of d'Alembert,

* See Article 23, Tome I., Traité de Mécanique, 2d ed., Paris, 1833.

† 'Théorie Analytique de la Chaleur,' Paris, 1822.

* Metz, 1826.

† Paris, 1829.

‡ Leipzig, 1837.

§ 'On a general method in dynamics.' *Philosophical Transactions*, 1834-35.

‖ For an account of these additions and for a complete list of papers bearing on the subject (up to 1857), one should consult the admirable report of Cayley on 'Recent progress in dynamics,' published in the Report of the British Association for the Advancement of Science for 1857.

by the equations of Lagrange, and by the equation of Hamilton. Each way has special advantages for particular applications, and together they may be said to condense into the narrow space of a few printed lines the net results of more than twenty centuries of effort in the formulation of the phenomena of matter and motion.

Such was the state of mechanical science when the great physical discovery of the century, the law of conservation of energy, was made. To give adequate expression to this law it was only necessary to recur to the Mécanique Analytique, for herein Lagrange had prepared almost all of the needful machinery. So well indeed were the ideas and methods of Lagrange adapted to this purpose that they have not only furnished the points of departure for many of the most important discoveries* of the present half century, but they have also supplied the criteria by means of which mechanical phenomena in general are most easily and effectively defined and interpreted.

Of the special branches of analytical mechanics which have undergone development during this century, by far the most important is that known as the theory of the potential function. This function first appeared in mathematical analysis in a memoir of Lagrange in 1777 † as the expression of the perturbative function, or force function. It next appeared in 1782 ‡ in a memoir by Laplace. In this memoir Laplace's equation § appears for the first time, being here expressed in polar coördinates. In 1787 * the same equation appears in the more usual form as expressed by rectangular coördinates.

Strange as it now seems when viewed by the light of this end of the century, nearly thirty years elapsed before Laplace's equation was generalized. Laplace had found only half of the truth, namely, that which applies to points external to the attracting masses.† Poisson discovered the other half in 1813.‡ Thus the honors attached to the introduction of this remarkable theorem are divided between them, and we now speak of the equation of Laplace and the equation of Poisson, though the equation of Poisson includes that of Laplace.

Next came the splendid contributions of George Green under the modest title of "An essay on the application of mathematical analysis to the theories of electricity and magnetism."§ It is in this essay that the term 'potential function' first occurs. Herein also his remarkable theorem in pure mathematics, since universally known as Green's theorem, and probably the most important instrument of investigation in the whole range of mathematical physics, made its appearance.

We are all now able to understand, in a general way at least, the importance of Green's work, and the progress made since the publication of his essay in 1828. But fully to appreciate his work and subsequent progress, one needs to know the outlook for the mathematico-physical sciences as it appeared to Green at this time, and to realize his refined sensitiveness in promulgating his discoveries.

* Especially those in the theories of electricity, magnetism, and thermodynamics.

† *Nouveaux Mémoires de l'Académie des Sciences et Belles Lettres de Berlin.* See also remarks of Heine, Handbuch der Kugelfunctionen, Band II., p. 342.

‡ *Paris Memoires* for 1782, published in 1785.

§ $$\Delta^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

$\Delta^2 V$ is called the Laplacian of V.

* *Paris Memoires* for 1787, published in 1789.

† That is, Laplace's equation is $\Delta^2 V = 0$, while Poisson's is $\Delta^2 V + 4\pi k\rho = 0$, V being the potential and ρ the density at the point (x, y, z), and k being the gravitation constant.

‡ Poisson's equation was derived in a paper published in *Nouveau Bulletin* * * * *Société Philomatique*, Paris, Dec., 1813.

§ Nottingham, 1828.

"It must certainly be regarded as a pleasing prospect to analysts," he says in his preface, "that at a time when astronomy from the state of perfection to which it has attained, leaves little room for further applications of their art, the rest of the physical sciences should show themselves daily more and more willing to submit to it. * * * Should the present essay tend in any way to facilitate the application of analysis to one of the most interesting of the physical sciences, the author will deem himself amply repaid for any labor he may have bestowed upon it; and it is hoped the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man, who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means, as other indispensable avocations which offer but few opportunities of mental improvement, afforded." Where in the history of science have we a finer instance of that sort of modesty which springs from a knowledge of things?

The completion of the potential theory, so far as it depends on the Newtonian law of the inverse square of the distance, must be credited to Gauss, though a host of writers has since contributed many valuable additions in the way of details. Early in the century Gauss had begun the study of the absorbing problems of the day, namely, problems of attractions and repulsions. The prevailing notion of mathematical physicists seems to have been that all mechanical phenomena may be attributed to attractions and repulsions between the ultimate particles of matter and the ultimate particles of 'fluids' associated with matter. The difficulties of action at a distance, without the aid of an intervening medium, happily, did not trouble them at that time; for who shall say that their labors would have been more fruitful if they had stopped to remove these difficulties? Gauss's first memoir in this field relates to the attractions of homogeneous ellipsoidal masses,* and dates from 1813. It was in this memoir that he published a number of the elegant theorems † which are now found in the elementary books on the theory of the potential function. In 1829 he published his theory of fluid figures in equilibrium,‡ and in 1832 there followed one of the most important papers of the century on the intensity of terrestrial magnetic force expressed in what we now call absolute units. § Six years later he published his wonderful theory of the earth's magnetism || and applied it to all existing observational data. This theory is a splendid application of the potential theory, and his entire investigation is one of the most beautiful and useful contributions to mathematical physics of the century. Well was he qualified, therefore, to complete the theory of the Newtonian potential function in the collection of theorems published in his memoir ¶ of

* 'Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum,' 1813. See Gauss's Werke, Band V., Göttingen, 1877.

† Especially the theorem giving the values of the surface integral

$$\int \frac{\cos (s, n)}{s^2} dS,$$

where dS is an element of any closed surface, s the distance from dS to any fixed point, and n indicates the normal to the surface at dS. This gave the key to the very important theorem of the surface integral of the normal acceleration, or

$$\int \frac{\partial V}{\partial n} dS.$$

‡ 'Principia generalia theoriæ figuræ fluidorum in statu æquilibrii,' 1829. Werke, Band V.

§ 'Intensitas vis magneticæ terrestris ad mensuram absolutam revocata.' Werke, Band V.

|| Allegemine Theorie des Erdmagnetismus. Werke, Band V.

¶ 'Allegemine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse das Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte.' Werke, Band V.

1840. This is still the fundamental memoir on the subject of which it treats, and must be regarded as one of the most perfect models of mathematical exposition. In respect to clearness and elegance, indeed, the works of Gauss are unsurpassed. "In his hands," as Todhunter has said,* "Latin and German rival French itself for clearness and precision." "Alles gestaltet sich neu unter seinen Händen," was the tribute † of Bessel; and the lapse of two generations has served only to increase admiration for the genius and industry which made Gauss one of the most conspicuous figures in the science of the nineteenth century.

The importance of the theory of the potential function when considered in its historical aspects is found to consist not so much in the rich harvest of results it has afforded in the field of gravitation, as in its direct bearing on the developments of other branches of mathematical physics. For the points of view and the analytical methods of the Newtonian function have been adapted and extended with brilliant success to the interpretation of almost all kinds of mechanical phenomena. Thus it has come about that we have now to deal with many kinds of potential, as logarithmic potential, velocity potential, displacement potential, electric potential, magnetic potential, thermodynamic potential, etc., each of which bears a more or less close mathematical analogy to the Newtonian function.

In the closing paragraph of his Exposition du Système du Monde, Laplace refers to the immense progress made in astronomy since the geocentric theory was displaced by the heliocentric theory of the solar system. This progress is specially remarkable when we consider that it depended on the discovery, so humiliating to man, of the relatively insignificant dimensions and inconspicuous rôle of our planet. But we agree with Laplace that "Les resultats sublimes auxquels cette découverte l'a conduit sont bien propres à le consoler du rang qu'elle assigne à la Terre, en lui montrant sa propre grandeur dans l'extrême petitesse de la base qui lui a servi pour mesurer les cieux." All astronomy is based on a knowledge of the size, the shape and the mechanical properties of the earth; and it is not surprising, therefore, that a large share of the mathematical investigations of the century should have been directed to the science of geodesy. Founded in the middle of the last century by Clairaut * and his contemporaries; recast by Laplace and Legendre † (1752–1833) in the early part of this century; systematized and extended to a remarkable degree by the German geodesists, led especially by the incomparable Bessel; ‡ this science has now come to occupy the leading position in point of perfection of methods and precision of results. So great, in fact, has been the growth of this science during the century that recent writers have found it desirable to subdivide the subject into two parts called mathematical geodesy and physical geodesy respectively, though both parts are nothing if not mathematical. §

In a former address I have considered somewhat in detail certain of the more

* History of the Theories of Attraction and Figure of the Earth, Vol. II., p. 235.

† In a letter to Olbers, 1818.

* Clairaut's work, Théorie de la Figure de la Terre, Paris, 1743, was the pioneer work in physical geodesy.

† The name of Legendre is famous in geodesy by reason of his beautiful theorem which makes the solution of a geodetic triangle almost as easy as the solution of a plane triangle.

‡ Bessel's contributions to astronomy and geodesy are collected in Abhandlungen von F. W. Bessel, herausgegeben von Rudolf Engelmann, in drei Bänden, Leipzig, Wilhelm Engelmann, 1875.

§ See, for example, Die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie von Dr. F. R. Helmert, Leipzig, B. G. Teubner, Teil I., 1880; Teil II., 1884.

salient mathematical problems which have arisen in the study of the earth;* and the present review may hence be restricted to a rapid résumé of the less salient but perhaps more recondite problems, and to the briefest mention of problems already discussed.

Adopting the convenient nomenclature of geologists, we may consider the earth as made up of four parts, namely, the atmosphere; the hydrosphere, the oceans; the lithosphere, or crust, and the nucleus. Beginning with the first of these we are at once struck by the fact that much greater progress has been made during the century in the investigation of the kinetic phenomena of the atmosphere than in the study of what may be called its static properties. Evidently, of course, the phenomena of meteorology are essentially kinetic, but it would seem that the questions of pressure, temperature and mass distribution of the atmosphere ought to be determined with a close approximation from purely statical considerations. This appears to have been the view of Laplace, who was the first to bring adequate knowledge to bear upon such questions. He investigated the terrestrial atmosphere as one might investigate the gaseous envelope of an unilluminated planet.† He reached the conclusion that the atmosphere is limited by a lenticular-shaped surface of revolution whose polar and equatorial diameters are about 4.4 and 6.6 times the diameter of the earth respectively, and whose volume is about 155 times that of the earth.‡ If this conclusion be true our atmosphere should reach out to a distance of about 26,000 miles at the equator and to a distance of about 17,000 miles at the poles. It does not appear, however, that Laplace attempted to assign the distribution of pressure and density, and hence total mass of the atmosphere within this envelope; and I am not aware that any subsequent investigator has published a satisfactory solution of this apparently simple problem.*

*On the Mathematical Theories of the Earth. Vice-presidential address before Section of Astronomy and Mathematics of the American Association for the Advancement of Science, 1889. *Proceedings of A. A. A. S.*, for 1889.

† Mécanique Céleste, Livre III., Chap. VII., and Livre X., Chapts. I.–IV.

‡ Laplace's equation to a meridian section of this envelope is

$$x^{-1}-x_0^{-1}+\tfrac{1}{2}ax^2\cos^2\phi=0,$$

where $x=r/a$, r being the radius vector measured from the center of the earth and a the mean radius of the earth; α is the ratio of centrifugal to gravitational acceleration at the equator of the earth; ϕ is geocentric latitude, and x_0 is the value of x for $\phi=\pi/2$.

The problem of the statical properties of the atmosphere may be stated in three equations, namely:

$$\Delta^2V+4\pi k\rho-2\omega^2=0,$$
$$dp=\rho dV,$$
$$p=f(\rho,\tau).$$

In these V is the potential at any point of the atmosphere; p, ρ, τ being the pressure, density and temperature at the same point; k is the gravitation constant; and ω is the angular velocity of the earth. The above equation of Laplace neglects the mass of the atmosphere in comparison with the mass of the rest of the earth. An essential difficulty of the problem lies in the unknown form of the function $f(\rho,\tau)$.

*I have sought a solution with a view especially to determine the mass of atmosphere. A class of solutions satisfying the mechanical conditions of the following assumptions has been worked out. Thus, assuming $p=c\rho^m$, which includes the adiabatic relation, $p=c\rho^{1.41}$, and the famous Laplacian relation, $\partial p/\partial\rho=2c\rho$; and the law of Charles and Gay-Lussac, $p=C\rho\tau$; there results

$$\frac{p}{p_0}=\left(\frac{Q}{Q_0}\right)^{\frac{m}{m-1}},\quad \frac{\rho}{\rho_0}=\left(\frac{Q}{Q_0}\right)^{\frac{1}{m-1}},\quad \frac{\tau}{\tau_0}=\frac{Q}{Q_0},$$

where $$Q=x^{-1}-x_0^{-1}+\tfrac{1}{2}ax^2\cos^2\varphi$$

defined above; Q_0 is the value of Q for $x=1$ and $\phi=\pi/2$; and p_0, ρ_0, τ_0 are the values of p, ρ, τ at the same point $(x=1,\ \phi=\pi/2)$.

Using the adiabatic law the above formula for ρ leads to a mass for the atmosphere of about 1/1200th of the entire mass of the earth. But since the adiabatic law gives too low a pressure, density and temperature gradient, this can only be regarded as an upper limit to the mass of the atmosphere. A lower limit of about 1/1000000th of the earth's mass is found by assuming that the mass of the atmosphere is equal to the mass of water or mercury which would give an equivalent pressure at the earth's surface.

On the other hand, the general character of the circulation of the atmosphere and the meteorological consequences thereof, have been brought within the domain of mathematical research, if they have not yet been wholly reduced to quantitative precision. The pioneer in this work was a fellow-countryman, William Ferrel (1817–1891),* who, like Green, came near being lost to science through the obscurity of his early environment. It is a curious though lamentable circumstance, illustrating at once the peculiar shyness of Ferrel and the proverbial popular indifference to discoveries which cannot be patented, that a man who had mastered the Principia and the Mécanique Céleste and who had laid the foundation of our theory of the circulation of the atmosphere, should have found no better medium for the publication of his researches than the semi-popular columns of a journal devoted to medicine and surgery. But such was the medium through which Ferrel's 'Essay on the Winds and Currents of the Ocean' appeared † in 1856. Since that time notable progress has been made at the hands of Ferrel, Helmholtz (1821–1894), Oberbeck, Bezold and others ;‡ so that we may entertain the hope that the apparently erratic phenomena of the weather will presently yield to mathematical expression, just as the similar phenomena of oceanic tides and terrestrial magnetism have already yielded to the power of harmonic analysis.

* For a biography and autobiographical sketch of Ferrel, and a list of his publications, see Biographical Memoirs of the National Academy of Sciences, Vol. III., pp. 265–309. Washington, 1895.

† In *Nashville Journal of Medicine and Surgery*, Oct. and Nov., 1856.

‡ Some of the most important papers and memoirs on this subject, collected and translated by Professor Cleveland Abbe, have been published by the Smithsonian Institution under the title 'The Mechanics of the Atmosphere.' Smithsonian Miscellaneous Collections, No. 843, Washington, 1891.

When we pass from the atmosphere to the hydrosphere, several questions concerning the nature and properties of their common surface, or what is usually called the sea surface, immediately demand attention. The most important of these are what may be distinguished as the static and the kinetic phenomena of the sea surface. Since tidal oscillations belong more properly to hydrokinetics, we may here confine attention to the static phenomena.

Starting from the datum plane fixed by Laplace, the most important contribution to the theory of physical geodesy since his time is the remarkable memoir of Sir George Gabriel Stokes 'On the Variation of Gravity at the Surface of the Earth.'* Adopting the hypothesis of original fluidity, or the more general hypothesis of a symmetrical arrangement of the strata of the earth, with increasing density towards the center, Laplace had shown that the acceleration of gravity in passing from the equator to the poles should increase as the square of the sine of the latitude.† This conclusion agreed well with the facts of observation ; and Laplace rested content in the opinion that his hypothesis was verified. But Stokes showed that the law of variation of the acceleration of gravity at the surface of the sea is wholly determined by that surface, regardless of the mode of distribution of the earth's mass. This, as we now see, of course, is a direct result of the theory of the potential function ; for the sea surface is an equipotential surface, and since it is observed to be closely spheroidal, the formula of Laplace follows independently of all hypothesis save that of the law of gravitation. But while Laplace's formula

* Read April, 1849. See Mathematical and Physcal Papers by G. G. Stokes, Cambridge University Press, 1883, Vol. II.

† Laplace's formula is $g = a + \beta \sin^2\phi$, where a is the value of g at the equator, β is a constant, and ϕ is the latitude of the place.

and the arguments by which he reached it throw no light on the distribution of the earth's mass, a slight extension of his methods gives a formula which shows that any considerable difference in the equatorial moments of inertia of the earth would produce a variation in the acceleration of gravity dependent on the longitude of the place of observation.* Thus it is possible by means of pendulum observations alone to reach the conclusion that the mass of the earth is very nearly symmetrically distributed with respect to its equator and with respect to its axis of revolution.

A question of great interest with which the acceleration of gravity at the sea surface is closely connected is that of the earth's mass as a whole. About two years ago I published a short paper which gives the product of the mean density of the earth and the gravitation constant in terms of the coefficients of Laplace's formula and the dimensions of the earth.† It was shown that this product can be easily computed from existing data to five significant figures with an uncertainty of only one or two units in the last figure; thus making it possible to obtain the mass of the earth to a like degree of precision if the constant of gravitation can be equally well determined. In a subsequent communication to this Society it was explained that the product in question is equal to 3π divided by the square of the periodic time of an infinitesimal satellite which would pass around the earth just grazing the equator if there were no atmosphere to impede its progress. The periodic time of such a satellite would be 1 hour, 24 minutes, 20.9 seconds. Attention is called to this subject with the hope that some mathematician may point out another possible relation between the gravitation constant and the mean density of the earth which can be accurately observed, or that some physicist may show how the gravitation constant can be measured directly with a precision extending to five significant figures.

The lithosphere is the special province of the geologist, and we may hence pass on to the nucleus, or chief part of the mass of the earth. Much time and attention have been devoted to the study of the important but intricate problems which the geometers of the early part of the century left to their successors. But while the obscurities and vagaries of our predecessors have been cleared away, it must be confessed that our improved mathematical apparatus has not brought us very far ahead of the positions of Laplace and Fourier as regards the constitution and properties of the nucleus. With respect to the law of the distribution of density in the nucleus it may be said that although Laplace's law* is probably not ex-

* See Helmert, Geodäsie, Band II., p. 74. The expression for the acceleration is

$$g = a + \beta \sin^2 \phi + \gamma \cos^2 \phi \cos 2\lambda,$$

where a, β, γ are constants, and ϕ, λ are latitude and longitude respectively; and the constant γ involves the difference of the equatorial moments of inertia as a factor.

† See *The Astronomical Journal*, No. 424. This product is expressed thus:

$$k\rho = \frac{2\pi}{T^2} + \frac{3(as_1 + \beta s_2)}{4\pi a\sqrt{1-e^2}};$$

wherein k is the gravitation constant, ρ is the mean density of the earth, T is the number of mean solor seconds in a sidereal day, a and β are the first two constants in the formula $g = a + \beta \sin^2 \phi + \gamma \cos^2\phi \cos 2\lambda$ for the acceleration of gravity at the sea surface in latitude φ and longitude λ; a is the half major axis and e is the eccentricity of the earth's spheroid; and

$$s_1 = \frac{1}{2}\left\{1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\},$$

$$s_2 = \frac{1-e^2}{4e^2}\left\{\frac{2e}{1-e^2} + \log\frac{1-e}{1+e}\right\}.$$

The resulting numerical value is

$$k\rho = 36797 \times 10^{-11}/(\text{second})^2.$$

* The Laplacian distribution of pressure, density, and potential in the earth are defined essentially (neglecting the effect of rotation) by the three following equations:

act it is yet quite as nearly correct as our observed information requires.*

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(To be concluded.)

THE POSITION THAT UNIVERSITIES SHOULD TAKE IN REGARD TO INVESTIGATION.†

WHAT position shall universities take with regard to investigation? When the honor was done me of asking me to take part in this discussion, my first thought, after the sensation of complacency at the compliment, was that there could hardly be a *discussion* where all held probably very nearly the same views, and that the great difficulty would be to say anything that would not be better said by another. Then as I began to think more carefully, I saw that the question was not, as I had at first imagined it to be, "what shall universities do to encourage those on their staffs to investigate?" It is far wider than that. It comprises a whole group of questions concerning which there may be every shade of opinion. So the more I thought, the more I admired the wisdom the committee had shown in their choice of a subject. Later still, it dawned upon me that surely it is a most satisfactory sign of progress that this Society should meet to discuss such a subject, with the conviction that, though without the shadow of a legal right to make claims, we are, nevertheless, sure of a sympathetic hearing from both universities and the public.

First of all let us consider the place of investigation in education, as a means of mental training, quite apart from any definite results. Surely this alone opens a wide field for one afternoon's ramble, in which there are diverging and recrossing paths enough to furnish us the surprises of unexpected partings and unhoped-for reunions.

I would here remark that perhaps some confusion is possible from different interpretations of the word 'investigation.' According to some it means simply practical work, object teaching, or, better still, object study. According to others it is the search for something new. With regard to the value of the former we are all pretty well agreed. We do not need to be told what an advance it is over the old way of learning the statements of others concerning matters well within the sphere of observation. It may sometimes be carried too far, but in view of its great usefulness we will not quarrel with a little abuse. With what is meant by the second interpretation the case is different. Excepting some singularly gifted natures, it does not, in my opinion, concern the student. The universal or even the very general application of this method is the result of an extreme reaction. It rests on a fallacy. Because investigation is a good thing, and worthy of encouragement, which all must admit, it is assumed to be good for all, and an accepted method of education, which conclusion I cannot adopt. It is for the beginner to learn what is worth learning in his particular field first of all. It is not easy in these days to learn all that is worth learning even in a very restricted department. To start on investigation with this only half-learned is a direct

$$\Delta^2 V + 4\pi k\rho = 0,$$
$$dp = \rho dV,$$
$$\frac{\partial p}{\partial \rho} = c\rho;$$

where p, ρ, V are the pressure, density, and potential at any point of the mass, k is the gravitation constant, and c is a constant securing the equality of the members of the last equation.

* With regard to what constitutes an adequate theory in any case, see an instructive paper by Dr. G. Johnstone Stoney on 'The kinetic theory of gas, regarded as illustrating nature.' *Proceedings, Royal Dublin Society*, Vol. VIII. (N. S.), Part IV., No. 45.

† Discussion before the American Society of Naturalists at the New Haven Meeting, December 25, 1899