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MSS. intended for publication and books, etc., intended for review should be sent to the responsible editor, Professor J. McKeen Cattell, Garrison-on-Hudson, N. Y.

REPORT ON PROGRESS IN NON-EUCLIDEAN GEOMETRY.

IT marks an epoch in the history of mathematics that at a meeting of a great Association for the advancement of science there should be presented by invitation a Report on non-Euclidean geometry.

Its two creators, Lobachévski, who misnamed it Imaginary Geometry, and Bolyai János, under the nobler name Science Absolute of Space, failed utterly while they lived, to win any appreciative attention for what is to-day justly honored as one of the profoundest advances of all time. The only recognition, the only praise of the achievement of Lobachévski ever printed in his lifetime was by Bolyai Farkas, the father of his brilliant young rival, and appeared in a little book with no author's name on the title page, and which we have no evidence that Lobachévski ever saw, a little book so rare that my copy is probably the only one on the Western Continent.

When after more than forty years they were rescued from oblivion by Baltzer and Hoüel in 1866, still envious time gave them back only with an aspersion against the genuineness of their originality. A cruel legend tarnished still their fame so long delayed, so splendidly deserved.

Even when their creation had reached the high dignity of being made the subject of courses of lectures for consecutive semesters at the University of Göttingen, yet on page 175 of the second impression of these lectures, 1893, we still find Felix Klein saying, "Kein Zweifel bestehen kann, dass Lobatscheffsky sowohl wie Bolyai die Fragestellung ihrer Untersuchungen der Gaussischen Anregung verdanken."

It is a privilege to begin my report by announcing the rigorous demonstration that this ungenerous legend is untrue. This point need not further delay us, since it has been treated by me at length in SCIENCE, N. S., Vol. IX., No. 232, pages 813–817, June 9, 1899.

What a contrast to the pathetic neglect of its creators, Lobachévski dying blind, unrecognized, without a single follower, Bolyai János dying of disgust with himself and the world, lies in the fact that less than a year ago our American magazine, the *Monist*, secured from the famous Poincaré, at great cost, a brilliant contribution to this now universally interesting subject, which I had the honor, through my friend T. J. McCormack, of reading in the original French manuscript.

This extraordinary paper, published only in English translation, appears in the *Monist*, Vol. 9, No. 1, Oct., 1898, pages 1–43. In the first section of his greatest work, Lobachévski says: "*Juxtaposition* (contact) is the distinctive characteristic of solids, and they owe to it the name *geometric solids*, when we retain this attribute, taking into consideration no others whether essential or accidental.

"Besides bodies, for example, also time, force, velocity are the object of our judgment; but the idea contained in the word juxtaposition does not apply thereto. In our mind we attribute it only to solids, in speaking of their composition or dissection into parts.

"This simple idea, which we have received directly in nature through the senses, comes from no other and consequently is subject to no further explanation. Two solids A and B, touching one another, form a single geometric solid C, in which each of the component parts A, B appears separate without being lost in the whole C. Inversely, every solid C is divided into two parts A and B by any *section* S.

"By the word section we understand here no new attribute of the solid, but again a juxtaposition, expressing thus the partition of the solid into two juxtaposed parts.

"In this way we can represent to ourselves all solids in nature as parts of a single whole solid which we call space."

Poincaré starts off somewhat differently. He says: "We at once perceive that our sensations vary, that our impressions are subject to change. The laws of these variations were the cause of our creating geometry and the notion of geometrical space.

"Among the changes which our impressions undergo, we distinguish two classes:

"(1) The first are independent of our will and not accompanied by muscular sensations. These are *external changes* so-called.

"(2) The others are voluntary and accompanied by muscular sensations. We may call these *internal changes*.

"We observe next that in certain cases when an external change has modified our impressions, we can, by voluntarily provoking an internal change, re-establish our primitive impressions. The external change, accordingly, can be *corrected* by an internal change. External changes may consequently be subdivided into the two following classes:

"1. Changes which are susceptible of being corrected by an internal change. These are *displacements*.

"2. Changes which are not so susceptible. These are *alterations*. An immovable being would be incapable of making this distinction. *Such a being, therefore, could never create geometry*, even if his sensations were variable, and even if the objects surrounding him were movable."

How like what Lobachévski said more than sixty years before: "We cognize directly in nature only motion, without which the impressions our senses receive are not possible. Consequently, all remaining ideas, for example, geometric, are created artificially by our mind, since they are taken from the properties of motion; and therefore space in itself, for itself alone, does not exist for us."

Poincaré continues: "*the aggregate of displacements is a group.*" At once rise before us the great names Riemann, Helmholtz, Sophus Lie. In fact Poincaré's next section is merely a restatement of part of Riemann's marvellous address, published 1867, on the hypotheses at the basis of geometry.

Again, though the work of Helmholtz did not contain the group idea, yet it had put the problem of non-Euclidean geometry into the very form for the instrument of Sophus Lie, who calls it the Riemann-Helmholtz Space-problem.

To the genius of Helmholtz is due the conception of studying the essential characteristics of a space by a consideration of the movements possible therein.

Felix Klein it was who first called the attention of Lie to this work of Helmholtz, before then unknown to Lie, and pointed out its connection with Lie's Theory of Transformation groups, inciting him to a group-theory investigation of the problem. In 1886 Lie gave briefly his weightiest results in a note: "Bemerkungen zu v. Helmholtz' Arbeit über die Thatsachen, die der Geometrie zu Grunde liegen," in the Berichte of the Saxon Academy, where, in 1890, he gave his completed work in two papers, 'Ueber die Grundlagen der Geometrie' (pp. 284–321, 355–418). The whole investigation published in Volume III. of his 'Theorie der Transformationsgruppen,' 1893, was in 1897 awarded the first Lobachévski Prize. Felix Klein declared that it excels all comparable works so absolutely that a doubt about the award could scarcely be possible. Lie gives two solutions of the problem. In the first he investigates in space a group possessing free mobility in the infinitesimal, in the sense, that if a point and any line-element through it be fixed, continuous motion shall still be possible; but if besides any surface element through the point and line-element be fixed, then shall no continuous motion be possible. The groups in tri-dimensional space possessing in a real point of general position this free mobility, Lie finds to be precisely those characteristic of the Euclidean and two non-Euclidean geometries. Strangely enough, for the seemingly analogous and simpler case of the plane or two-dimensional space these are not the only groups. There are others where the paths of the infinitesimal transformations are spirals. Without the group idea, Helmholtz had reached this reality, and as a consequence concluded that also to characterize our tri-dimensional spaces a new condition, a new axiom, was needed, that of *monodromy*. It is one of the most brilliant results of Lie's second solution of the space problem, that starting from transformation-equations with three of Helmholtz's four assumptions, he proves that the fourth, the famous 'Monodromie des Raumes,' is, in space of three dimensions, wholly superfluous. What a demonstration of the tremendous power of Lie's Group Theory! Lie's method in general, as it appears in the Berichte, is the following:

Consider a tri-dimensional space, in which a point is defined by three quantities x, y, z.

A movement is defined by three equations: $x_1 = f(x, y, z)$; $y_1 = \varphi(x, y, z)$; $z_1 = \psi(x, y, z)$.

By this transformation an assemblage, A, of points (x, y, z) becomes an assemblage, A', of points (x_1, y_1, z_1).

This represents a movement which

changes A to A'. Now make, in regard to the space to be studied, the following assumptions:

(B) In reference to any pair of points which are moved, there is *something* which is left unchanged by the motion. That is, after an assemblage of points, A, has been turned by a single motion into an assemblage of points, A', there is a certain function, Ω, of the coördinates of any pair of the old points (x_1, y_1, z_1), (x_2, y_2, z_2) which equals that same function, Ω, of the corresponding new coördinates (x_1', y_1', z_1'), (x_2', y_2', z_2'); that is
$\Omega(x_1', y_1', z_1', x_2', y_2', z_2') = \Omega(x_1, y_1, z_1, x_2, y_2, z_2)$. This *something* corresponds to the generalized idea of distance interpreted as independent of measurement by superposition of an unchanging sect as unit for length. Moreover assume:

(C) If one point of the assemblage is fixed, every other point of this assemblage, *without any exception*, describes a surface (a two-dimensional aggregate). When two points are fixed, a point in general (exceptions being possible) describes a curve (a one-dimensional aggregate). Finally, if three points are fixed, all are fixed (exceptions being possible). Then Lie proves exhaustively that the group consists either of all motions of Euclidean space or of all motions of non-Euclidean space.

The result is a remarkable one, demonstrating that the group of Euclidean motions and the group of non-Euclidean motions are, in tri-dimensional space, the only groups in which exists in the strict sense of the word free mobility. Thus free motion in the strict meaning of the word can happen in three and only three spaces, namely, the traditional or Euclidean space, and the spaces in which the group of movements possible is the projective group transforming into itself one or the other of the surfaces of the second degree $x^2 + y^2 + z^2 \pm 1 = 0$.

To the fundamental assumption which completely characterizes these three groups, Lie gives also this form:

"If any real point y_1^0, y^0_2, y_3^0 of general position is fixed, then all real points x_1, x_2, x_3, into which may still shift another real point x_1^0, x_2^0, x_3^0, satisfy a real equation of the form:

"$W(y_1^0, y_2^0, y_3^0;\ x_1^0, x_2^0, x_3^0;\ x_1, x_2, x_3) = 0$, which is not fulfilled for $x_1 = y_1^0$, $x_2 = y_2^0$, $x_3 = y_3^0$, and which represents a real surface passing through the point x_1^0, x_2^0, x_3^0.

"About the point y_1^0, y_2^0, y_3^0 may be so demarcated a triply extended region, that on fixing the point y_1^0, y_2^0, y_3^0, every other real point x_1^0, x_2^0, x^0_3 of the region can yet shift continuously into every other real point of the region, which satisfies the equation $W = 0$ and which is joined to the point x_1^0, x_2^0, x_3^0 by an irreducible continuous series of points."

It is a satisfaction to the world of science that Lie's vast achievements were recognized while he lived. Poincaré accepts and expounds his doctrine, saying in the article already mentioned: "The axioms are not analytical judgments *a priori;* they are conventions. * * * Thus our experiences would be equally compatible with the geometry of Euclid and with a geometry of Lobachévski which supposed the curvature of space to be very small. We choose the geometery of Euclid because it is the simplest.

"If our experiences should be considerably different, the geometry of Euclid would no longer suffice to represent them conveniently, and we should choose a different geometry."

When on November 3, 1897, the great Lobachévski prize was awarded to Lie, three other works were given honorable mention. The first of these is a thesis on non-Euclidean geometry by M. L. Gérard, of Lyons. Lovers of the non-Euclidean geometry are naturally purists in geometry, and keenly appreciate Euclid's using solely such figures as he has rigorously constructed. They understand that problems of construction play an essential part in a scientific system of geometry. Far from being solely, as our popular text-books suppose, practical operations, available for the training of learners, they have in reality, as Helmholtz declares, the force of existential propositions. Therefore is evident the high import of Gérard's work to establish the fundamental propositions of non-Euclidean geometry without hypothetical constructions other than the two assumed by Euclid: 1. Through any two points a straight line can be drawn; 2. A circle may be described from any given point as a center with any given sect as radius. Gérard adds explicitly the two assumptions: 3. A straight line which intersects the perimeter of a polygon in a point other than one of its vertices intersects it again; 4. Two straights, or two circles, or a straight and a circle, intersect if there are points of one on both sides of the other.

Upon these four hypotheses, perfecting a brilliant idea of Battaglini (1867), Gérard establishes the relations between the elements of a triangle.

Lobachévski never explicitly treats the old problems changed by transference into the new geometric world, such as "Through a given point to draw a parallel to a given straight"; nor yet the seemingly impossible problems now in it capable of geometric solution, such as "To draw to one side of an acute angle the perpendicular parallel to the other side"; "To square the circle."

These would be sought in vain in the two quarto volumes of Lobachévski's collected works. Bolyai János, in his all too brief two dozen pages, gives solutions of them startling in their elegance.

But in establishing his theory, he uses, for the sake of conciseness, the principle of continuity even more freely than does Lobachévski.

Gérard, in the second part of his memoir, gives the elements of non-Euclidean analytic geometry, and in the third part, a strict treatment of equivalence.

Even Euclid, in proving his I., 35, "Parallelograms on the same base, and between the same parallels, are equal to one another," does not show that the parallelograms can be divided into pairs of pieces admitting of superposition and coincidence. He uses rather the assumption explicitly set forth by Lobachévski, "Two surfaces are equal when they are sums or differences of congruent pieces." But Creswell in his *Treatise of Geometry*, showed how to cut the parallelograms into parts congruent in pairs. The same can be done for Euclid I., 43, "The complements of the parallelograms, which are about the diagonal of any parallelogram are equal." Hence, we may use the definition: Magnitudes are equivalent, which can be cut into parts congruent in pairs. This method I applied to the ordinary Euclidean geometry in my Elementary Synthetic Geometry before the appearance of Gérard's work, where it is extended to the non-Euclidean.

Regarding the first assured construction of Euclid and Gérard: "A straight line can be drawn through any two points," W. Burnside has given us a charming little paper in the *Proceedings of the London Mathematical Society*, Vol. XXIX., pp. 125–132 (Dec. 9, 1897), enitled 'The Construction of the Straight Line Joining Two Given Points.' Euclid's postulate implies the use of a ruler or straight-edge of any required finite length. The postulate is clearly not intended to apply to the case in which the distance between the two points is infinite. In fact, Euclid I., 31, gives a compass and ruler construction for the line when one of the points can be reached while the other cannot. The other exceptional case when neither point can be reached, *i. e.*, when two given points are the points at infinity on two non-parallel lines, is not dealt with by Euclid.

In elliptic space any one point can be reached from any other by a finite number of finite operations. The line joining two given points can therefore be always constructed with the ruler alone. In hyperbolic space, if we deal with projective geometry, we must assume that *every* two straight lines in a plane determine a point. When the two straight lines are non-intersectors, the point can neither be a finite point nor a point at infinity. Such a point is termed an 'ideal' point. The problem of constructing the straight line joining two given points involves therefore three further cases; namely, (IV) that in which one of the points is a finite point and the other an ideal point; (V) that in which one is a point at infinity and the other an ideal point; (VI) that in which both points are ideal points.

It is a pleasure to signal the appearance, within the past year, of the second volume of the exceedingly valuable work of Dr. Wilhelm Killing, 'Einführung in die Grundlagen der Geometrie,' (Paderborn, 1898).

With Killing's name will be associated the tremendous difference living geometers find between the properties of a finite region of space, and the laws which pertain to space as a whole. Of the word *direction* he says "it can only be given a meaning when the whole theory of parallels is already presupposed."

The pseudo-proof of the parallel postulate still given in current text-books, for example, by G. C. Edwards in 1895, Killing calls the Thibaut proof, saying that it has especial interest because its originator, who was professor of mathematics at Göttingen with Gauss, published the attempt at a time, 1818, when Gauss had already called attention to the failure of attempts to prove this postulate, and declared that we had not progressed beyond where Euclid was 2000 years before.

But Killing is here in error when he supposes Thibaut the originator of this popular pseudo-proof. It was given in 1813 by Playfair in his edition of Euclid, in a Note to I., 29. It was very elegantly shown to be a fallacy by Colonel T. Perronet Thompson, of Queen's College, Cambridge, in a remarkable book called 'Geometry without Axioms,' of which the third edition is dated 1830, a book seemingly unknown in Germany, since Engel and Staeckel copy from Riccardi the title (with the mistake 'first books' for 'first book') under the date 1833, which is the date of the fourth edition.

Killing has won an important place by investigating the question, what varieties of connection of space are compatible with the different elemental arcs of constant curvature. Riemann, Helmholtz and Lie consider only a region of space, and give analytic expressions for the vicinity of a point. If this region be extended, the question is, what kind of connection of space can result.

Killing shows there are different possibilities, really a series of topologically different forms of space with Euclidean, Lobachévskian, Riemannian geometry in the bounded, simply connected region.

The germinal idea is due to Clifford, who, in an unprinted address before the Bradford meeting of the British Association (1873), 'On a surface of zero curvature and finite extent,' and also by a remark in his paper 'Preliminary sketch of biquaternions,' called attention to a recurrent surface in single elliptic space, which has everywhere zero for measure of curvature, yet is nevertheless of finite area.

Similarly complete universal spaces are found of zero or negative measure of curvature, which nevertheless are only of finite extent. Since there is no way of proving that the whole of our actual space can be moved in itself in ∞^6 ways, it may possibly be, after all, one of these new Clifford spaces. Free mobility of bodies may only exist while they do not surpass a certain size.

Killing devotes an interesting section, over seven pages, to Legendre's definition of the straight line as the shortest distance between two points. He emphasizes three principle reasons why this is inadmissible. These are (*a*) since the possibility of measurement for all lines is presumed beforehand, which is not allowable; (*b*) since before the execution of the measurement there must be a measuring standard, but this is first given by the straight line; (*c*) since the existence of a minimum is not evident, on the contrary can be demanded only as an assumption.

The first objection was always conclusive, yet it strengthens every day, for our new mathematics knows of lines, real boundaries between two parts of the plane, to which the idea of length is inapplicable.

Under the title 'Universal Algebra,' one would scarcely look for a treatise on non-Euclidean geometry. Yet the first volume of Whitehead's admirable work (Cambridge, 1898, pp. 586) devotes more than 150 pages to an application of Grassmann's Calculus of Extension to hyperbolic, elliptic, parabolic spaces. So devoted is he, that we find him saying: "Any generalization of our space conceptions, which does not at the same time generalize them into the more perfect forms of hyperbolic or elliptic geometry, is of comparatively slight interest." He emphasizes the fact that the three-dimensional space of ordinary experience can never be proved parabolic. "The experience of our senses, which can never attain to measurements of absolute accuracy, although competent to determine that the space-constant of the space of ordinary experience is greater than some large value, yet cannot, from the nature of the case, prove that this space is absolutely Euclidean."

From the many important contributions by Whitehead may be singled out as especially timely his development of a theorem of Bolyai János to which F. S. Macauly called especial attention in the second of his able articles entitled, John Bolyai's 'Science Absolute of Space' (*The Mathematical Gazette,* No. 8, July, 1896, pp. 25–31; No. 9, October, 1896, pp. 49–60). Macauly says, p. 53, "Finally follows a theorem (§ 21), which is, undoubtedly, the most remarkable property of hyperbolic space, that the sum of the angles of any triangle formed by L-lines on an F-surface is equal to two right angles. On this theorem Bolyai remarks: (Halsted's Bolyai, 4th Ed., p. 18), 'From this it is evident that Euclid's Axiom XI., and all things which are claimed in geometry, and plain trigonometry hold good *absolutely* in F, L-lines being substituted in place of straights. Therefore, the trigonometric functions are taken here in the same sense (are defined here to to have the same values) as in Σ (as in Euclidean geometry); and the periphery of the circle, of which the L-form radius $= r$ in F, is $= 2\pi r$, and likewise the area of circle with radius r (in F) $= \pi r^2$ (by π understanding half the periphery of circle with radius 1 in F, or the known 3.1415926 * * *).'"

Whitehead, in his Universal Algebra, § 262, recurs to this important point, saying: "The idea of a space of one type as a locus in space of another type, and of dimensions higher by one, is due partly to J. Bolyai, and partly to Beltrami. Bolyai points out that the relations between lines formed by great circles on a two-dimensional limit-surface are the same as those of straight lines in a Euclidean plane of two dimensions. Beltrami proves by the use of the pseudosphere, that a hyperbolic space of any number of dimensions can be considered as a locus in Euclidean space of higher dimensions. There is an error, popular even among mathematicians, misled by a useful technical phraseology, that Euclidean space is in a special sense flat, and that this flatness is exemplified by the possibility of a Euclidean space containing surfaces with the properties of hyperbolic and elliptic spaces. But the text shows that this relation of hyperbolic to Euclidean space can be inverted. Thus no theory of the flatness of Euclidean space can be founded on it." Whitehead has since followed up his point in a very important and powerful paper in the *Proceedings of the London Mathematical Society*, Vol. XXIX., pp. 275–324, March 10, 1898, entitled 'The Geodesic Geometry of Surfaces in non-Euclidean Space.' He there says, "The relations between the properties of geodesics on surfaces and non-Euclidean geometry, as far as they have hitherto been investigated, to my knowledge, are as follows:

"It has been proved by Beltrami that the 'geodesic geometry' of surfaces of constant curvature in *Euclidean* space is the same as the geometry of straight lines in planes in elliptic or in hyperbolic space, according as the curvature of the surface is positive or negative.

"The geometry of great circles on a sphere of radius ρ in elliptic space of 'space-constant' γ is the same as the geometry of straight lines in planes in elliptic space of space-constant $\gamma \sin \frac{\rho}{\gamma}$.

"The geometry of great circles on a sphere of radius ρ in hyperbolic space of 'space-constant' γ is the same as the geometry of straight lines in planes in elliptic space of space-constant $\gamma \sin\mathrm{h}\, \frac{\rho}{\gamma}$.

"The geometry of geodesics (that is, lines of equal distance), on a surface of equal distance, σ, from a plane in hyperbolic space of space-constant γ, is the same as that of straight lines in planes in hyperbolic space of space-constant $\gamma \cos\mathrm{h} \frac{\sigma}{\gamma}$.

"Finally, the geometry of geodesics (that is, limit-lines), on a limit surface in hyperbolic space—which may be conceived either as a sphere of infinite radius or as a surface of equal, but infinite, distance from a plane—is the same as that of straight lines in planes in Euclidean space.

"The preceding propositions are due directly, or almost directly to John Bolyai, though, of course, he only directly treats of hyperbolic space.

"From the popularization of Beltrami's results by Helmholtz, and from the unfortunate adoption of the name 'radius of space curvature' for γ (here called the space-constant), many philosophers, and, it may be suspected from their language, many mathematicians, have been misled into the belief that some peculiar property of flatness is to be ascribed to Euclidean space, in that planes of other sorts of space can be represented as surfaces in it. This idea is sufficiently refuted, at least as regards hyperbolic space, by Bolyai's theorem respecting the geodesic geometry of limit surfaces. For a Euclidean plane can thereby be represented by a surface in hyperbolic space.

"It is the object of this paper to extend and complete Bolyai's theorem by investigating the properties of the general class of surfaces in any non-Euclidean space, elliptic or hyperbolic, which are such that their geodesic geometry is that of straight lines in a Euclidean plane.

"Such surfaces are proved to be real in elliptic as well as in hyperbolic space, and their general equations are found for the case when they are surfaces of revolution.

"In hyperbolic space, Bolyai's limit-surfaces are shown to be a particular case of such surfaces of revolution. The surfaces fall into two main types; the limit surfaces form a transition case between these types. In elliptic space there is only one type of such a surface of revolution.

"The same principles would enable the problem to be solved of the discovery in any kind of space of surfaces with their 'geodesic' geometry identical with that of planes in any other kind of space."

So that which Macauly designated as 'undoubtedly the most remarkable property of hyperbolic space' has been by Whitehead not only generalized for hyperbolic space but extended to elliptic space.

Bolyai János seemed fully to realize the weight, the scope, the possibilities, the meaning of his discovery. He returns to it in § 37, where he uses the proportionality of similar triangles in F to solve an essential problem in S (hyperbolic space). Then he adds: "Hence, easily appears (L-lines being given by their *extremities alone*) also fourth and mean *terms* of a proportion can be found, and all geometric constructions which are made in Σ in plano, in this mode can be accomplished in F *apart from Axiom XI.*" The italics are Bolyai's, yet I find that they have not been reproduced in my published translation (the only one in English), nor in Frischauf's German, nor in Hoüel's French, nor in Fr. Schmidt's Latin text, nor in Suták's Magyar. Whitehead's researches will remind us all how great a thing it was to have reached the whole Euclidean system entirely apart from any parallel-postulate. It is a pleasure to be able to state that this was also done by Lobachévski. It is explicitly given in his first published work 'O nachalah geometri' (1829). 'Noviya nachala geometri' (1835), devotes to it Chapter VIII.

It is also at this point, so striking as pure mathematics, that general philosophy finds itself involved. Killing, Klein, and in general the German writers, distinctly draw back from any philosophical implications. The whole matter, however, has been opened in 'An Essay on the Foundations of Geometry,' by Hon. Bertrand A. W. Russell, Fellow of Trinity College, Cambridge (1897), who has had the good fortune to be the very first to set forth the philosophical importance of von Staudt's pure projective geometry, which in its foundation and dealing with the qualitative properties of space involves no reference to quantity. I discussed this point more than twenty years ago in the *Popular Science Monthly*, *à propos* of Spencer's classification of the Abstract Sciences.

In a note to the first edition of his classification of the sciences (omitted in the second edition), Spencer says, "I was ignorant of this as a separate division of mathematics, until it was described to me by Mr. Hirst. It was only when seeking to affiliate and define 'Descriptive Geometry' that I reached the conclusion that there is a negatively-quantitative mathematics as well as a positively-quantitative mathematics." As explanatory of what he wishes to mean by negatively-quantitative, we quote from his Table I.: "Laws of Relations, that are Quantitative (Mathematics), Negatively: the terms of the relations being definitely-related sets of positions in space, and the facts predicted being the absence of certain quantities ('Geometry of Position')." He also says: "In explanation of the term 'negatively-quantitative,' it will be sufficient to instance the proposition that certain three lines will meet in a point, as a negatively-quantitative proposition, since it asserts the absence of any quantity of space between their intersections. Similarly, the assertion that certain three points would always fall in a straight line is 'negatively-quantitative,' since the conception of a straight line implies the negation of any lateral quantity or deviation." But Sylvester has said of this very proposition that it "refers solely to position, and neither invokes nor involves the idea of quantity or magnitude."

"Projective Geometry proper," says Russell, "does not employ the conception of magnitude."

Now it is in metrical properties alone that non-Euclidean and Euclidean spaces differ. The distinction between Euclidean and non-Euclidean geometries, so important in metrical investigations, disappears in projective geometry proper. Therefore projective geometry deals with a wider conception, a conception which includes both, and neglects the attributes in which they differ. This conception Mr. Russell calls 'a form of externality.' It follows that the assumptions of projective geometry must be the simplest expression of the indispensable requisites of all geometrical reasoning.

Any two points uniquely determine a line, *the straight*. But any two points and their straight are, in pure projective geometry, utterly indistinguishable from any other point pair and their straight. It is of the essence of *metric* geometry that two points shall completely determine a spatial *quantity, the sect* (German, strecke). If Mr. Russell had used for this fundamental spatial magnitude this name, or any name but 'distance,' his exposition would have gained wonderfully in clearness. It is a misfortune to use the already overworked and often misused word 'distance' as a confounding and confusing designation for a sect itself and also the measure of that sect, whether by superposition, ordinary ratio, indeterminate as depending on the choice of a unit; or by projective metrics, indeterminate as depending on the fixing of the two points to be taken as constant in the varying cross ratios.

That Mr. Russell's chapter 'A Short History of Metageometry,' contains all the stock errors in particularly irritating form, and some others peculiarly grotesque, I have pointed out in extenso, in Science, Vol. VI., pp. 478–491. Nevertheless the book is epoch-making. It finds "that projective geometry, which has no reference to quantity, is necessarily true of any form of externality. In metrical geometry is an empirical element, arising out of the alternatives of Euclidean and non-Euclidean space."

One of the most pleasing aspects of the universal permanent progress in all things non-Euclidean is the making accessible of the original masterpieces.

The marvellous '*Tentamen*' of Bolyai Farkas, as Appendix to which the 'Science Absolute' of Bolyai János appeared, a book so rare that except my own two copies, I know of no copy on the Western Continent, a book which has never been translated, a field which has lain fallow for sixty-five years, is now being re-issued in sumptuous quarto form by the Hungarian Academy of Sciences. The first volume appeared in 1897, edited, with sixty-three pages of notes in Latin, by König and Réthy of Budapest. Professor Réthy, whom I had the pleasure of meeting in Kolozsvár, tells me the second volume is in press, and he is working on it this summer.

Bolyai Farkas is the forerunner of Helmholtz, Riemann, Lie, though one would scarcely expect it from the poetic exaltation with which he begins his great work. "Lectori salutem! Scarce superficially imbued with the rudiments of first principles, of my own accord, without any other end, but led by internal thirst for truth, seeking its very fount, as yet a beardless youth, I laid the foundations of this 'Tentamen.'

"Only fundamental principles is it proposed here so to present, that Tyros, to whom it is not given to cross on light wings the abyss, and, pure spirits, glad of no original, to be borne up in airs scarce respirable, may, proceeding with firmer step, attain to the heights.

"You may have pronounced this a thankless task, since lofty genius, above the windings of the valleys, steps by the Alpine peaks; but truly everywhere are present gordian knots needing swords of giants. Nor for these was this written.

"Forsooth I wish the youth by my example warned, lest having attacked the labor of six thousand years, alone, they wear away life in seeking now what long ago was found. Gratefully learn first what predecessors teach, and after forethought build. Whatever of good comes, is antecedent term of an infinite series."

His analysis of space starts with the principle of continuity: spatium est quantitas, est continuum (p. 442). This Euclid had used unconsciously, or at least without specific mention; Riemann and Helmholtz consciously. Second comes what he calls the *axiom of congruence*, p. 444, § 3, "corpus idem in alio quoque loco videnti, quæstio succurrit: *num loca ejusdem diversa æqualia sint?* Intuitus ostendit, æqualia esse."

Riemann: "Setzt man voraus, dass die Körper unabhängig von Ort existieren, so ist das Krümmungsmass überall constant." See also the second hypothesis of Helmholtz.

Third, any point may be moved into any other; the free mobility of rigid bodies. If any point remains at rest any region in which it is may be moved about it in innumerable ways, and so that any point other than the one at rest may recur. If two points are fixed, motion is still possible in a specific way. Three fixed points not costraight prevent all motion (p. 446, § 5).

Thus we have the third assumption of Helmholtz, combined with his celebrated principle of Monodromy.

Bolyai Farkas deduces from these assumptions not only Euclid but the non-Euclidean systems of his son János, referring to the approximate measurements of astronomy as showing that the parallel postulate is not sufficiently in error to interfere with practice (p. 489). This is just what Riemann and Helmholtz afterward did, only by casting off also the assumption of the infinity of space they got also as a possibility for the universe an elliptic geometry, the existence of a case of which independently of parallels was first proven by Bolyai János when he proved spherics independent of Euclid's assumption. So if Sophus Lie had ever seen the 'Tentamen,' he might have called his great investigation the Bolyai-Farkas Space Problem instead of the Riemann-Helmholtz Space Problem.

The first volume of the 'Tentamen' as issued by the Hungarian Academy does not contain the famous appendix. But in 1897, Franz Schmidt, that heroic figure, ever the bridge between János and the world, issued at Budapest, the Latin text of the Science Absolute, with a biography of Bolyai János in Magyar, and a Magyar translation of the text by Suták József.

Strangely enough, though the Appendix had been translated into German, French, Italian, English, and even appeared in Japan, yet no Hungarian rendering had ever appeared. It was Franz Schmidt who placed the monument over the forgotten grave of János, only identified because there still lived a woman who had loved him. Now in this Magyar edition he rears a second monument. The introduction by Suták is particularly able.

The Russians have honored themselves by the great Lobachévski Prize; why does not that glorious race, the Magyars, do tardy justice to their own genius in a great Bolyai Prize?

One other noble thing the Hungarian Academy of Science has just achieved, the publication in splendid quarto form of the correspondence between Gauss and Bolyai Farkas: (Briefwechsel zwischen Carl Friedrich Gauss und Wolfgang Bolyai). It was again Franz Schmidt who, after long endeavors, at last obtained this correspondence from the Royal Society of Sciences at Göttingen, where Bolyai had sent the letters of Gauss at his death. The Correspondence is fitly edited by Schmidt and Staeckel. It gives us a romance of pure science. Gauss was the greater mathematician ; Bolyai the nobler soul and truer friend. On April 10, 1816, Bolyai wrote to Gauss giving a detailed account of his son János, then fourteen years old ; and unfolding a plan to send János in two years to Göttingen, to study under Gauss. He asks if Gauss will take János into his house, of course for the usual remuneration, and what János shall study meanwhile. Gauss never answered this beautiful and pregnant letter, and never wrote again for sixteen years ! Had Gauss answered that letter Göttingen might now perhaps have to boast a greater than Gauss, for in sheer genius, in magnificent nerve, Bolyai János was unsurpassable, as absolute as his science of space. But instead, he joined the Austrian army, and the mighty genius which should have enriched the transactions of the greatest of learned societies with discovery after discovery in accelerating quickness, preyed instead upon itself, printing nothing but a brief two dozen pages.

Almost to accident the world owes the admirable volumes in which Staekel and Engel contribute such priceless treasures to the non-Euclidean geometry. An Italian Jesuit, P. Manganotti, discovered that one of his order, the Italian Jesuit Saccheri, had already in 1733 published a series of theorems which the world had been ascribing to Bolyai. Thereupon, in 1889, E. Beltrami published in the *Atti della Reale Accademia dei Lincei*, Serie 4, Vol. V., pp. 441–448, a note entitled 'Un Precursore italiano di Legendre e di Lobatschewski,' giving extracts from Saccheri's book which abundantly proved the claim of Manganotti.

In the same year, 1889, E. d'Ovidio, in the *Torino Atti*, XXIV., pp. 512–513, called attention to this note in another entitled, Cenno sulla Nota del prof. E. Beltrami: "Un Precursore, etc.," expressing the wish that P. Manganotti would by a more ample discussion rescue Saccheri's work from unmerited oblivion. Staeckel says the thought then came to him, whether Saccheri's work were not a link in a chain of evolution, the genesis of the non-Euclidean geometry.

In 1893, at the International Mathematical Congress at Chicago, in the discussion which followed my lecture, 'Some Salient Points in the History of Non-Euclidean and Hyper-Spaces,' wherein I gave an account of Saccheri with description of his book and extracts from it, Professor Klein, who had never before heard of Saccheri, and Professor Study, of Marburg, mentioned that there had recently been brought to light an old paper of Lambert's anticipating in points the non-Euclidean geometry, and named in connection therewith Dr. Staeckel. I at once wrote to him and published in the *Bulletin of the New York Math. Soc.*, Vol. III., pp. 79–80, 1893, a note on Lambert's non-Euclidean geometry, mentioning Staeckel's purpose to republish Lambert's paper in the Abhandlungen of the Leipziger Gesellschaft der Wissenschaften. But after this, in January, 1894, Staeckel formed the plan to make of Saccheri and Lambert a book, and associating with him his friend Friedrich Engel, they gave the world in 1895, 'Die Theorie der Parallellinien, eine Urkundensammlung zur Vorgeschichte der nichteuklidischen Geometric.' Strengthened by the universal success of this book, they planned two volumes in continuation. Staeckel takes the volume devoted to Bolyai János and his father. It is to begin with a more complete life of the two than has yet appeared, of course from material furnished largely by Franz Schmidt.

Then follows the 'Theoria parallelarum'

of Bolyai Farkas, interesting as proving that in 1804 Gauss was still under the spell of Euclid.

Then is to follow the Latin text of the immortal Appendix with a German translation. Next comes in German translation selections from the 'Tentamen.' The book concludes with the geometric part of 'Kurzer Grundriss,' the only one of the Bolyai's works printed originally in German. This volume is nearly published and may be expected in a few weeks. The volume undertaken by Engel has just appeared (1899). It is a German translation of Lobachévski's first published paper (1829), 'On the Principles of Geometry,' and also of his greatest work, 'New Elements of Geometry, with Complete Theory of Parallels.' Only from the 'New Elements' can any adequate idea be obtained of the height, the breadth, the depth of Lobachévski's achievement in the new universe of his own creation.

Of equal importance is the fact that Engel's book gives to the world at last a complete, available text-book of non-Euclidean geometry. There is no other to compare with it.

For the history of non-Euclidean geometry we have the admirable Chapter X., of Loria's pregnant work, 'Il passato ed il presente delle principali teorie geometriche.' This chapter cites about 80 authors, mostly of writings devoted to non-Euclidean geometry.

In my own 'Bibliography of hyper-space and non-Euclidean geometry,' in the *American Journal of Mathematics* (1878), I gave 81 authors and 174 titles. This, when reprinted in the Collected Works of Lobachévski (Kazan, 1886), gives 124 authors and 272 titles.

Roberto Bonola has just given in the *Bollettino di Bibliografia e Storia della Scienze Matematiche* (1899), an exceedingly rich and valuable 'Bibliografia sui Fondamenti della Geometria in relazione alla Geometria Non-Euclidea,' in which he gives 353 titles.

This extraordinary output of human thought has henceforth to be reckoned with. Hereafter no one may neglect it who attempts to treat of fundamentals in geometry or philosophy.

GEORGE BRUCE HALSTED.

AUSTIN, TEXAS, Aug. 14, 1899.

BOTANY AT THE COLUMBUS MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

SECTION 'G' was attended by a large number of Botanists and the meeting was in every way pleasurable and profitable.

On Monday afternoon Charles R. Barnes gave the vice-presidential address in Botanical Hall of the Ohio State University, to a large and appreciative audience. His theme was the 'Progress and Problems in Vegetable Physiology,' and the address has been published in full in SCIENCE.

During each of the succeeding four days, two sessions were held and thirty-three papers were read and discussed. Wednesday was made a Memorial Day to Sullivant and Lesquereux; the exercises are described below by Mrs. Britton.

Among the items of business transacted by Section 'G' may be mentioned that which related to the publication of the card index of American Botany, and an expression of high appreciation of the appointment of an eminent physiological chemist in the Division of Vegetable Pathology and Physiology, United States Department of Agriculture.

The authors of papers and an outline of the more important points are herewith presented:

'The Fertilization of Albugo bliti,' by F. L. Stevens, Chicago, Ill.

The paper presented the results of two year's research on the development of the