ter for the employer to be liberal in estimating the time-rate rather than with the premium-rate. Excessive premium-rates are apt to result in too large expectations to be fully met in the long run. From one-half to one-third the saving are usual premium-rates, and probably one-third to the workman and two-thirds to the firm best brings out a permanent and satisfactory adjustment which, if found inequitable, can generally be easily readjusted to a correct figure. In one machine-tool works the premium-rate is thirty-six per cent. and is found satisfactory to both sides. The higher premium-rates, however, should be paid for manual labor, as in blacksmithing, and the lower to power-tool work, as at the lathe or the planer or the milling machine. Undoubtedly every establishment, and every department of labor, from floor-sweeping to book-keeping, has its own peculiar best rate. In all cases the result may be expected to be a largely increased output of the works, a greatly increased earning power on the part of the men, and decreased costs of production with increased dividend-paying power for the holders of the capital. "Wisely administered, the plan will do more to settle the wages-question than anything else that has been suggested," and the wages-question is to-day the burning question in the economics of manufacturing.

R. H THURSTON.

SCIENTIFIC BOOKS.

Analytic Functions. Introduction to the Theory of Analytic Functions. By J. HARKNESS and F. MORLEY. London, Macmillan & Co. 1898 8vo. Pp. xvi + 336.

The appearance of the present work is a very pleasant sign to friends of the modern school of mathematics in England and America. It indicates that the movement which set in some years past with us in this direction has been steadily growing; that the theory of functions is no longer the property of a few bold and restless minds, but has already descended to the masses. The present work may very happily serve as a text or reference book to a first course on the theory of functions in the senior class of any of our better universities. The theory of functions of a complex variable may be viewed from two standpoints. One was taken by Cauchy and Riemann; the other by Weierstrass. The methods of Cauchy and Rieman are more natural and intuitive; those of Weierstrass more abstract and lend themselves more easily to a rigorous treatment of the subject. The authors have chosen the methods of Weierstrass.

Roughly speaking, the subjects treated in the first 100 pages fall under two heads:

1. The geometric representation of complex numbers, the conformal representation afforded by

$$y = \frac{ax + b}{cx + d}$$

and the first properties of rational functions.

2. Topics which lie at the foundation of the calculus.

The treatment of the first group of subjects is admirable. In regard to the second is seems to us that the authors have attempted the impossible. The theory of function in common with the calculus rests on certain notions, such as that of number, limit, continuity, extremes of functions, etc. These subjects are very imperfectly treated in English works on the calculus, and our authors have thus found it advisable to give some account of them in the present volume. The amount of space at their disposal was very limited, and they have, therefore, been obliged to be excessively concise. This has been carried to such an extent in the chapter on number, Chapter I., that the subject, so it seems to us, will be utterly incomprehensible to the student.

We cannot understand why, if it is worth while to say anything about irrational numbers, the arithmetical operations upon them are passed over in absolute silence. Until the terms sum, product, etc., are defined they have no meaning.

Chapter VI., which treats of limits and continuity, suffers severely on account of the brevity of Chapter I. In this chapter it is important to establish the existence of certain numbers. The arguments cannot have much meaning to the student until the material of Chapter I. has been grasped, and this seems out of the question.

Before leaving this section we call attention to a curious break. On page 48 complex functions of a real variable are differentiated and integrated. This certainly is illogical until such operations have been defined. We are tempted to believe that the beauties of this chapter will fall very flat with the average student. If the geometrical theory of the logarithm is to appeal to him, what is stated here so rapidly should be given with leisure and detail.

The next 60 pages, Chapters VIII.–XII., deal with infinite series, and so lead us to Weierstrass's conception of analytic functions. This, as is known, depends on infinite series ascending according to integral powers of $(x-a)$. The treatment here is very superior—the authors show a masterly grasp of the subject. A short chapter on the analytic theory of the exponential and logarithmic function now follows.

Chapters XIV. and XV., pp. 178–209, turn again to the general theory. Singular points are discussed, and Weierstrass's decomposition of a function into prime factors is deduced. Application is made to show that

$$\sin \pi x = \pi x \Pi (1 - x^2/n^2). \qquad n = 1, 2, \cdots, \infty.$$

The consideration of the zeros gives at once

$$\sin \pi x = x e^{G(x)} \Pi (1 - x^2/n^2).$$

The determination of the integral transcendental function G is singularly difficult. It seems a pity that the method invented for Cauchy for the same purpose and which may easily be made rigorous is to-day quite neglected. By this method G is readily found.

With Chapter XVI., which treats of integration, we arrive at the starting point of the Cauchy-Rieman theory. It seems to us that our authors have not maintained the high ideals here as well as elsewhere. In a passage, pp. 11, 12, we read: "But in using geometric intuitions * * * we must emphasize one lesson of experience; that the intuitional method is not in itself sufficient for the superstructure. It has been found that only by the notion of number * * * can fundamental problems be solved. If, however, we are prepared to replace when occasion arises these geometric intuitions * * * then and only then is the use of geometry thoroughly available." It is true that the authors *here* speak of points, distances and angle only, but these remarks apply with equal cogency, as they will be the first to admit, to all geometric intuitions when used in analysis. We are, therefore, surprised to find the obscure notion of curve, of its length, of a closed curve, of a region, etc., freely used without any attempt to put them on a number basis. Such statements as that on p. 189, viz: that a circuit divides the entire plane into two regions will certainly embarrass the authors to prove in its generality. Again, on p. 213, we see the authors implicitly define the length of a curve C to be $\int_C |dx|$. This definition differs from the one given our text-books, viz.: $\int dx \sqrt{1 + f'(x)^2}$. As our authors propose to use a broader definition than usual, it seems only fair that they state this to the reader. Still a more serious objection is to be urged to their procedure. It results in stating Cauchy's fundamental theorem and other important theorems of this chapter without any restriction regarding the path of integration. This seems to us like talking of infinite series without bothering ourselves about convergence.

Chapter XVII. brings a brief discussion of Laurent's and Fourier's series. Then follow two excellent chapters on the elliptic functions. These are followed by two chapters or about 30 pages devoted to Algebraic functions and Riemann surfaces.

It appears to us that the fictitious number and point ∞ has been treated too hurriedly. These notions are very important and also difficult for the student to master. Our authors have followed the usual custom of disposing of them with a few words here and there. We believe the custom of introducing the number ∞ is bad. The theory of functions of a complex variable is a theory of two very special real functions of two real variables. In the theory of functions of real variables the number ∞ does not exist. It seems to us that its introduction can only produce confusion and embarrassment.

It is not a number ∞ we are ever concerned with. When we say $w(a) = \infty$ we really mean $\lim |w(z)| = \infty$, $x = a$. Again when we ask how does $w(z)$ behave for $z = \infty$ we really mean how does $w(1/\varsigma)$ behave in the *vicinity* of $\varsigma = o$ where $\varsigma = 1/z$. Thereby ς is never required to assume the value of o. On using the sphere instead of the plane we get the punktierte Kugel. The missing point we can supply or not at our option. In any case no number shall correspond to it. We firmly believe that the easy intuitional way of treating ∞ in the function theory of a complex variable must be modified as here indicated.

The last chapter is devoted to a brief *aperçu* of the function theory from the standpoint of Cauchy and Riemann. We cannot appreciate the difficulties mentioned in § 164 as underlying the definition of a function from the Cauchy-Riemann standpoint. They seem to us to be due to the belief on the part of the authors that we must take the whole z-plane into our definition from this point of view. Such is not the case. As a domain D for the variable z we take any point multiplicity consisting only of *interior* points. If it be possible to pass from any point of D to any other of it along a continuous curve $x = \phi(t)$, $y = \psi(t)$ we say D is a simple domain. Otherwise D is composed of simple domains $D = D_1 + D_2 + \ldots$ To get a synectic function $w(z)$ for D we take two single valued functions $u(x,y)$, $v(x,y)$ defined over D and such that for every point in D they have a total differential and satisfy the equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In any one of these simple regions as R_α, $w(z)$ can be developed into an integral positive power series. The analytic function $f(z)$ obtained from one of these elements is identical with $w(z)$. There certainly is no reason to suppose that $f(z)$ when continued into another region R_β should be identical with $w(z)$ in this region. This seems to answer all the objections in I and II of this article. Indeed, the advantage seems to be decidedly on the side of Cauchy, for exactly one of the points urged against Cauchy's theory is now without force, while it is, indeed, an important matter from Weierstrass' standpoint. This, in the author's words, is: "That Cauchy's definition implies in various ways a considerable preliminary grasp of the logical possibilities attached to the study of singular points." From our standpoint we fix in advance the domain D; it has no more singular points than we choose to assign. Not so with the analytic function. Here an element is given, one singular point must lie on its circle of convergence. Where the others are is a subject of further study.

We cannot see the difficulty mentioned under III. It is, ineeed, an interesting matter to know 'the irreducible minimum of conditions to impose on $w(z)$,' but it seems to us nowise necessary. It suffices that we know the necessary and sufficient conditions in order that $w(z)$ can be developed according to Taylor's Theorem. This we know and we have taken them into our definition of $w(z)$. It may be interesting to remark, however, that these conditions are already known, as will appear in a remarkable paper of E. Goursat shortly to be published.

We close, congratulating the authors for writing a work which we believe will prove an excellent aid to acquire some of the essentials of the theory of function. We should have preferred to see the two theories of Cauchy and Weierstrass blended together into an organic and indivisible whole. Although these two theories grew up quite distinct, they have already been welded into *one* greater and more powerful theory. It is only the purist who still tenaciously clings to the methods of Weierstrass. It seems, therefore, very desirable to us that an introductory work should be written more in accordance with this fact.

JAMES PIERPONT.
YALE UNIVERSITY, March, 1899.

A Handbook of Metallurgy. By DR. CARL SCHNABEL. Translated by HENRY LOUIS. New York, The Macmillan Company. Two volumes, medium 8vo. Total pages, 1608. Illustrated. Volume 1, copper, lead, silver, gold. Volume 2, zinc, cadmium, mercury, bismuth, tin, antimony, arsenic, nickel, cobalt, platinum, aluminum. Price, $10.00.

The author states in the preface that, while many exhaustive works have appeared on the