I have here made no mention of the work that has been done on the various theories of the mutual actions at a distance of current elements, as these are thoroughly dealt with in J. J. Thomson's admirable British Association report on electrical theories in 1885. I have thus, in a very brief and unsatisfactory manner, merely touched upon some of the principal points of the development of the theory of electricity, and traced its gradual but unceasing progress from the hands of the giants of the old days to those of the new.

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THE LIMITATIONS OF THE PRESENT SOLUTION OF THE TIDAL PROBLEM.

THAT which is new in science is always interesting. But it is well at times to let the old and well-tried pass in review before us, to plan renewed attacks upon the unknown, in the light of the elements of strength or weakness found in different portions of the army of known facts and principles, and with respect to the stubborness of the resistance which has been encountered by attacks upon different parts of the unknown. A helter-skelter attack may perhaps produce more interesting and more surprising results than a well planned campaign, but the latter would be expected to furnish the more important results. The purpose of this paper is not to state anything new, but to point out a very weak point in tidal theory, a point which it is important to have strengthened, and of which the strengthening is apt to lead to a decided advance in our knowledge of the subject.

The thesis which I submit is that the present theory of the tides upon the earth when used to explain those tides, or to predict their occurrence at a particular point, furnishes very little except the *periods* of the separate harmonic, or invariable, components of the tide. It does not furnish the times of occurrence of the tides, that is, the epochs of the components, nor does it furnish the range of the tide as defined by the amplitudes of the harmonic components.

This thesis may be exhibited in concise form by writing the algebraic expression for the height of the tide referred to mean sea level at any instant at a given point

$$h = A_1 \cos(\alpha_1 t + \beta_1) + A_2 \cos(\alpha_2 t + \beta_2) \ldots\ldots$$

Each term of this expression indicates one of the harmonic components of the tide. From pure theory, reasoning from the known motions of the Moon, Sun and Earth and the Newtonian law of gravitation, it has been shown that if certain definite values be assigned to the quantities $\alpha_1, \alpha_2, \ldots$ (fixing the periods of the separate terms), that each term truly represents one of the invariable components of the tide. Here, after *merely* fixing the periods of the separate components the contribution of tidal theory ends, and the work of direct observation at the particular station under consideration begins. The values of A_1, A_2, defining the amplitudes of the separate components and the range of the composite tide, and of β_1 $\beta_2 \ldots$ fixing the epochs of the separate components and the time of the tide, must be derived directly from observations at the particular station in hand. Their values cannot, at present, be assigned even approximately from theory.

If we look at the *actual* tidal problem presented to us by Nature the reason why theory furnishes so little will become evident. In approaching the actual problem let us begin by considering a simple ideal problem which can be and has been successfully handled, and then introduce into this problem, one by one, the *actual* conditions which modify it. In doing so we may, to a certain extent, follow the historic order of tidal research.

There are tides produced by both the Moon and the Sun. To avoid circumlocutions we will deal explicitly with the lunar tides only.

We may start with the problem of determining the shape of the free ocean surface on the supposition that the Earth consists of a rigid nucleus completely covered with a great and uniform depth of water and upon the supposition that the Moon moves in a circular orbit in the plane of the Earth's equator at such a rate that one face of the Earth is always presented to the Moon, just as the Moon now always presents the same face to us. Roughly speaking, the ocean in this case would become an ellipsoid with its major axis in the line joining the centers of the Earth and Moon, and this tide, if we may call it so, is the so-called static or equilibrium tide. This problem has been completely solved.

As a next step toward the actual problem, let the Moon be supposed to remain in a circular orbit in the plane of the Earth's equator, but let its period in that orbit be twenty-seven days, as at present it is. The Earth now presents different parts to the Moon in rapid succession as the Earth rotates once on its axis in each solar day. At once the problem becomes much more difficult than before ; for the effects of viscosity of the water, of its inertia and of friction against the ocean bottom combine to reduce the height of the wave, to modify the shape of the wave, to cause its crest to lag behind the Moon. The problem is now that of dealing with a *forced* wave—a much more difficult one than the static problem just outlined, and more difficult than that of dealing with a free wave. But it is still a problem which has been so thoroughly investigated that there is comparatively little hope for a new worker to extend our knowledge much along this line.

The introduction of the actual Moon with its rapid changes in declination and distance, and its variable motion in right ascension, in the place of the fictitious Moon we have been considering, makes the problem heavier, but not essentially much more difficult.

From the theory of wave motions in liquids it has been shown that for a wave, such as the tidal wave, of which the length from crest to crest is great in comparison with the depth of the water, the rate of progress of the wave is connected with the depth by the law $V = \sqrt{gh}$, in which V is the velocity of progress of the wave, g the acceleration due to gravity and h the depth of the water. According to this law a free tidal wave would succeed in keeping pace with the Moon in its apparent progress around the Earth only in case the assumed uniform depth of the water is greater than thirteen miles. If the uniform depth be assumed less than thirteen miles the old wave will, in effect, be continually being lost in the race and a new wave be continually being built up in front of it. The tide of our previous problem will be still further reduced in range and modified in shape and will lag still further behind the Moon, which produces it. The problem is still tractable, though exceedingly difficult, and still falls in the category of thoroughly investigated problems.

Now let the problem become nearer like that of Nature by supposing the shape of the surface of the solid portion of the Earth to be just what it is in fact, an irregular succession of great continental elevated areas, great oceanic basins, mountain ranges, broad valleys, great plateaus, etc. Let the problem still differ from that of Nature by supposing that the water level is just high enough to cover the summit of Mt. Everest, so that the whole Earth is covered with depths varying from zero, at Mt. Everest, to ten miles, on a few small areas at the deepest portions of the oceans. If, under these conditions, an attempt is made to follow the history of the wave in its westward progress, the problem will at once be found to be exceedingly difficult, if not intractable, though the problem is still much simpler than that presented to us by Nature.

The depths being everywhere less than thirteen miles, the wave will at no point be able to keep pace with the Moon. The apparent rate of progress of the Moon over the Earth's surface is about seventeen miles per minute. The rate of progress of the wave will be about fifteen miles per minute where the depth is ten miles, and less than five miles per minute where the depth is not greater than one mile.

Let an attempt be made to trace a tidal wave westward, starting from the 180th meridian in the Pacific. The northern portion of the wave will be confronted by the full width of both Asia and Europe; the middle portion will have deep water across the Indian Ocean, but will be obliged to cross submerged Africa before reaching the Atlantic; while the southern portion will have unobstructed deep water to and beyond the Cape of Good Hope. This southern portion, rounding the Cape of Good Hope, will necessarily be propagated northward up the Atlantic, as well as westward, for the middle portion, travelling in shallower water, will not yet have reached the Atlantic. The middle and northern portions of the wave will have acquired an irregular front on account of the various depths traversed by different portions. This Cape wave, as it progressed northward up the Atlantic valley, would combine at an angle in some complex fashion with the irregular wave emerging gradually from the shoals of Africa, and finally from Europe. In short, even with the conditions so far imposed, it would be exceedingly difficult, if not impossible, with our present knowledge, to compute the wave to be found in the Atlantic, to say nothing of the additional complexity produced when this already complicated wave crossed the submerged Americas.

Still we are dealing with a problem very much simpler than that of Nature. In the actual case the Moon would outstrip the wave in its westward progress, and then the direct effect of its attraction would be to tend to tear down the old wave and build up a new one in advance of it. This action would bring about radical modifications in the wave, even within one trip around the globe.

Moreover, the variations in depth produce other effects upon the wave fully as important as the one just considered. The increased friction in small depths tends to reduce the height of the wave. On the other hand, when a wave passes from deep to shallow water there is a tendency for the wave to attain a much greater height than before, since nearly the same amount of kinetic energy must be concentrated in a much smaller amount of water. So a tidal wave continually varies between wide limits in amplitude, as well as in its rate of propagation, in a way that is exceedingly difficult, to say the least, to compute. These variations in range produce obvious difficulties in such a case as that just suggested, in which waves from different regions coalesce.

As one more step in making our assumed problem approach the actual problem, let the water surface, which has been supposed to be at the level of the summit of Mt. Everest, subside to its present actual position. One-fourth of the earth's surface becomes dry land. There is now in the problem all the previous intricacies, and in addition a new set of difficulties, arising from the fact that the oceans are bounded by irregular shores, from which the tidal wave is *reflected* to a greater or less extent at every point of contact. The *actual* tidal waves may as properly be compared to a choppy sea, such as may be seen along the docks of a crowded port, as to a regularly progressive wave passing to westward around the earth, as pictured in the minds of most people.

Tracing the actual tidal wave as best we can by direct observation, we find that by starting at a point off the west coast of South Africa the wave is separated into two waves, one of which goes westward and the other eastward. If we follow that portion of the wave which has the least obstructed path, passing across the open Pacific to the southward of Australia and to the Cape of Good Hope, we find that after *thirty-six* hours it has not yet circumnavigated the globe, but instead has just reached the neighborhood of Cape Horn and is there apparently lost in a collision with an *eastward-bound* tidal wave, which started in the Pacific, off the west coast of South America, twenty-four hours later than the wave which we have followed. Looking to other portions of the oceans we find that the tidal wave moves northward rather than westward over the whole Atlantic and a part of the Pacific—in fact, on about one-third of the total ocean surface; and that the progress has a decided *eastward* tendency in at least two large areas, in the Arctic ocean north of Europe, and in the Pacific off the west coast of South America.

If we examine the relative amplitudes of the harmonic components of the tide at different stations we shall find further strong evidence of the radical modifications made in the astronomical tide by the influence of shores and bottom. As a typical case we may note that in the Atlantic the semi-diurnal components are large as compared with the diurnal components, so that there are always two tides of nearly the same height per day, whereas in the Pacific the amplitude of the diurnal components approaches that of the semi-diurnal components and in some cases exceeds it, and as a result the two Pacific tides of the same day are in general of decidedly different heights and in certain extreme cases but *one* tide occurs per day. In one of the extreme cases, at Batavia, Java, the component which has *one* high water per *sidereal* day is more than six times as large as the component having two high waters per lunar day, although in the astronomical tide the latter predominates over all others.

Out of the conflict with the shores and bottom the astronomical tide preserves only its periods, and hence in making tidal predictions at a given station theory can furnish us nothing but said periods. In the conflict no period is lost, though the amplitudes corresponding to certain periods may be greatly decreased or increased, and no new periods are known to be produced save certain multiple periods, or overtones, so to speak, produced by friction, and an annual period fixed probably in the main by meteorological causes.

When the prediction is made it applies to *one point* only, the point at which the observations were made, and with our present lack of ability to predict the effect of boundaries upon the range, the shape, and the rate of progress of the tidal wave that prediction can be extended even by careful study of charts but a very few miles from the stations of observation before acquiring large errors.

The effects of the boundaries—bottom and sides—can best be studied in bays and rivers, in bodies of water in which observations are available at many points, and in which the direct effect of the Moon and Sun is small as compared with that of the wave transmitted from the ocean. The tide tables and charts issued by the Coast and Geodetic Survey furnish many fine opportunities to study this problem. For example, the charts issued by that Survey give complete and detailed information as to soundings in the Chesapeake Bay and its tidal tributaries, and the tide tables give complete harmonic data for Old Point Comfort, Baltimore and Washington, and data as to time and mean range of tide for forty-nine other points on Chesapeake Bay, and for twenty-eight points along the Potomac River, to say nothing of sixty-one points on other tributaries of the bay. For such a region the investigator has an ample collection of facts to be used in proving or disproving any theory which he may formulate.

I am inclined to think that whoever successfully attacks this problem will use a graphic, or partially graphic method, plotting his results step by step upon the chart. In any wholly analytic method it will be especially difficult to take sufficiently into account the configuration of the bottom and shore.

In conclusion, I submit that to solve this boundary problem is to make an immense stride in our knowledge of the tides, a stride corresponding to a half century of ordinary progress; that it is in this line that our ignorance of the tides is most dense; that the facts are at hand for the investigation, and that, judging from the literature of the tides, this is, *comparatively speaking*, an unworked portion of the field. Along this line considerable pioneer work has been done, especially along purely mathematical lines, but the new comer will find neither a long series of failures to discourage him by indicating that the problem is intractable, nor a long series of successes to discourage him by making it appear that there is little opportunity to advance beyond what has already been done by others.

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It may seem that in this paper some attention should be paid to the fact that theory furnishes the relative amplitudes of certain harmonic components; that, in particular, theory indicates that certain relations exist between the relative amplitudes and the mass of the Moon, and that this theory has been born out by the fact that said mass has been computed with a high degree of accuracy from tidal observations.

It should, however, be kept clearly in mind that only the *relative* amplitudes are concerned in the computation of the Moon's mass. Further, the mass of the Moon as deducted from observations at a single tidal station is often largely in error. An accurate determination of the mass is obtained only when the results of observations at many stations are combined.

There is a decided significance, in the present connection, in the conclusions reached by two investigators who have carefully studied this phase of the tidal problem. Professor Ferrel, after a prolonged consideration of the matter, concludes that, to secure a better determination of the Moon's mass from the tides, a special study of 'shallow water components' should be made. In other words, the effects of friction due to the boundaries must be studied. Professor Harkness, in deriving the Moon's mass from tidal observations,* gives all stations equal weight, though the length of the series of observations varies at the different stations from one to nineteen years, on the ground that 'the accidental errors at any station are generally small as compared with those due to constant causes.' He indicates in the context that these 'constant courses' are constant for each point, but variable in passing from point to point along a coast; in other words, they are due to the local peculiarities of the boundaries.

JOHN F. HAYFORD.

GEOMETRICAL OPTICAL ILLUSIONS.

DURING the last few years the subject of Optical Illusions has been receiving a degree of attention that may well be called remarkable. Both popular and scientific articles have been written, so that the general public, as well as the specialist, is well

* On the Solar Parallax and its Related Constants, Wm. Harkness, pp. 119–120.