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MSS. intended for publication and books, etc., intended for review should be sent to the responsible editor, Prof. J. McKeen Cattell, Garrison-on-Hudson, N. Y.

*THE MATHEMATICAL THEORY OF THE TOP.**

LOOKING over such famous old books as Montmort's 'Analyse des jeux de hasard' or Moivre's 'Doctrine of Chances' one regrets that so much excellent mathematics should have been wasted on games most of which are wholly obsolete. Coriolus in his 'Jeu de billard' (1835) fared better, for the game is still very much alive and its dynamical terrors unsubdued. In even greater measure is this true of the top. The top has been everybody's toy and must, therefore, at one time or another have piqued everybody's curiosity. Lagrange, Poinsot, Jacobi, not to mention other great names, have in their turn paid tribute; yet the top may be set spinning to-day, unhampered by a completed theory to account for its evolutions.

Among recent contributions we may refer in particular to Professor A. G. Greenhill's† noteworthy papers, in which the algebraically accessible or pseudo-elliptic cases, such in which the integrations are possible in terms of circular functions, are worked out in full. Physicists will be grateful to Professor Greenhill for the concrete exhibition given of this complex motion. The

* Lectures delivered on the occasion of the sesquicentennial celebration of Princeton University, by Felix Klein, pp. 1–74, edited by Professor H. B. Fine. New York, Charles Scribner's Sons, 1897.

† Greenhill: Applications Elliptic Functions, Proc. Lond. Math. Soc., 1895, 1896; *Engineering*, July, 1896.

unique method of presentation adopted—all the curves being worked out in form of stereoscopic diagrams—endows his results with an objective reality; and when one remembers that these complex curves reach only especially simple cases of gyroscope motion, one may get some notion of the difficulty of the problem involved.

Turning now, from Greenhill's necessarily cumbersome equations for the approachable part of the problem of rotation, to Klein's little book, one is astonished in finding the most general aspects of the subject treated almost without computation and in so little space. This astonishment, however, is in a manner relieved on learning that the discussion remains formal throughout, that much of it is epitomized, many proofs sketched in, and that the reader is supposed to be thoroughly versed not only in dynamics, but familiarly conversant with the theory of complex variables, with elliptic integrals and functions particularly in reference to their derivation from ϑ and σ-functions, their generalization in terms of automorphic functions, and to be as well read as possible in the geometry of hyperspace. The reviewer, who makes no special pretense to these accomplishments, has taken up Klein's remarkable book, since it professedly appeals to physicists and has groaned through it. He ought, therefore, at the outset to confess to a feeling of hostility because of its unbending mathematical aloofness. In a book with a professed missionary purpose it is not unreasonable to expect just a little condescension in favor of the kind of mathematics with which physicists are, as a rule, more familiar. Judicious annotation either on the part of Professor Klein himself or by Professor Fine would have speeded the propagandist. I doubt whether everybody will 'at once' recognize the elliptic integrals of pages 28, 29 as being normals of the third type, particularly when the notation of Legendre and Jacobi is different. It would have cost but little to give the expanded form of the σ-function. If the reviewer is not incorrect, Weierstrass's original notation was in terms of Abelian functions. The tremendous development of elliptic functions is out of proportion with their application to natural phenomena. Meeting them rarely one forgets them. Memory peters out like the infinite series of a ϑ-function. Mathematicians will do well to observe that a reasonable acquaintance with theoretical physics in its present stage of development, to mention only such broad subjects as electricity, elastics, hydrodynamics, etc., is as much as most of us can keep permanently assimilated. It should also be remembered that the step from the formal elegance of theory to the brute arithmetic of the special case is always humiliating, and that this labor usually falls to the lot of the physicist.

To return from this paroxysm to the splendid research under discussion, let us note first that Klein begins his analysis with the top spinning on a sharp frictionless pivot, so that a simple point in the axis (not the center of mass) is fixed. To this special case the first three lectures are devoted. In the fourth the restriction is cut loose. Klein's method is to consist in a far-sighted choice of coordinates, and the first lecture is, therefore, a comparison of available systems with their mutual transformations. The Cartesian definition of three movable in terms of three fixed coordinates with a common origin in the fixed point, by the 9 direction cosines considered as functions of time, is first taken up. The corresponding transformation scheme is thereafter expressed in terms of Euler's ϑ, φ, ψ, parameters; in terms of the rotational or quaternion parameters, and finally in terms of Klein's new parameters, which are introduced as follows: x, y, z, and X, Y, Z, being the coordinates of given points on a fixed and a movable sphere, respectively, each of radius r and in congruence, the variables ζ and Z defined by the ratios

$$\zeta=\frac{x+iy}{r-z}=\frac{r+z}{x-iy},\quad Z=\frac{X+iY}{r-Z}=\frac{r+Z}{X-iY}$$

will be parameters each of which determines a point on the fixed and movable spheres, respectively. The unique advantage of these non-symmetrical parameters is that when the movable sphere (supposed fixed in the body) rotates, the relation of the parameters ζ and Z is a linear equation of the form

$$\zeta=\frac{\alpha Z+\beta}{\gamma Z+\delta}$$

where α and δ, γ and β are conjugate imaginaries. These quantities connected by the equation $\alpha\delta-\beta\gamma=1$, together with ζ, are used as variables specially adapted for treating the top problem. Hence a scheme of orthogonal substitution and a direct expression of the new parameters in terms of the Eulerian and rotational parameters is fully developed. The lecture closes with an even broader interpretation of ζ for the case when α and γ, β and δ are not conjugate, and time (t) for convenience in the theory of functions is also considered complex.

Starting on more familiar ground, the second lecture begins with a direct attack of the problem of rotation of a body (top) about a point other than its center of mass. Klein uses the expression for kinetic and potential energy in terms of Eulerian speed coordinates, the three corresponding Lagrangian equations of motion and the law of the conservation of energy to reduce the rotation to the following succinct specifications: Let ϑ, φ, ψ be the Eulerian coordinates and put $\cos\vartheta=u$. Let U be a polynomial of the third degree in u, involving besides only integration constants l, n, h, and the (maximum and therefore constant) static moment of the top with respect to the fixed point. Then

$$t=\int\frac{du}{\sqrt{U}},\quad \varphi=\int\frac{n-lu}{1-u^2}\frac{du}{\sqrt{U}},$$

$$\psi=\int\frac{l-nu}{1-u^2}\frac{du}{\sqrt{U}},$$

so that the motion is completely given (Lagrange) in terms of quadratures. Unfortunately, however, these integrals are elliptic and, except in the special cases worked out by Professor Greenhill, do not admit of algebraic treatment, while the 2d and 3d integrals are, beyond this, complex in type. Jacobi, to whom the introduction of elliptic functions is due, was thus able to make an immense stride forward by expressing the Lagrangian integrals u, φ, ψ, and therefore the equivalent cartesian direction cosines, as (one-valued) ϑ-functions of time; but while the direction cosines thus become much simpler time functions than the integrals, they are far more complicated than Klein's parameters α, β, γ, δ. It is the object of the remainder of Klein's brilliant research to show that these quantities are the simplest possible elliptic time functions compatible with the conditions of the problem.

Riemann's conformal representations are naturally selected as the appropriate method of treatment. The first integral (t) is approached by mapping out $\sqrt{U}$ on the plane of complex u. The surface obtained is two-leaved, consisting of two positive and two negative distinct half sheets which cross along segments of the real axis between the 1st and 2d root of cubic U, the 3d root and infinity.

A corresponding conformal representation is now made on the plane of complex time, defined by the first integral above. It is shown that as u moves through the real axis, in the u plane, t for a single half sheet of the $\sqrt{U}$ surface describes a rectangle in the t plane, whose position and sides (periods) are determinate when the time integral is made definite. Four adjacent and congruent rectangles in the t plane correspond to the four half sheets of the Riemann surface. Finally for any march of u around the segments between successive roots of U, t receives a constant increment, such that the complete image in the t plane covers the whole infinite t surface with congruent adjacent rectangles which nowhere overlap. Hence the important conclusion is accentuated that whereas for each point u there correspond an infinite number of values of time (t), for each value of t there corresponds but one value of u, and hence u like $\sqrt{U}$ are single-valued, doubly-periodic elliptic time functions.

Klein next takes up the relations of φ and ψ to t, a problem much more complex but one in which he scores his most signal triumph. Introducing his own parameters α, β, γ, δ, already defined in terms of Euler's coordinates, Klein obtains normal integrals of the third type without further reduction, while the four logarithmic discontinuities are assignable, one each to log α, log β, log γ, log δ, with a common logarithmic discontinuity at $u = \infty$. The transformation thence to exponentials (α, β, γ, δ) is equivalent in Klein's interpretation to a passage from elliptic integrals to elliptic functions, and now he is able to avail himself of the quotient of two σ-functions (each of which contains null-points only), together with an exponential time factor to fully express his parameters. They severally vanish for $u = \pm 1$ and became ∞ for $t = 0$, one in each parallelogram of periods. Finally the 9 direction cosines known in terms of α, β, γ, δ are, therefore, also expressed in term of quotients of σ-functions.

Having thoroughly unveiled the character of his parameters α, β, γ, δ, Klein proceeds with their application. The Z pole of the moving sphere is preferably selected for tracing top curves. At this point $Z = \infty$, and, therefore, the paths on the fixed sphere become $\zeta = \alpha/\gamma$. Hence ζ too is at once expressible as a single quotient of single valued σ-functions, together with an exponential time factor. An essential simplification has thus been achieved over all preceding methods. Hermite in his treatment of the stereographic projection of the Z pole needed functions as complex as products of Klein's functions, while even in the hands of Jacobi the first degree of complexity reached only the specialized case of a Poinsot motion, *i. e.*, rotation relative to a fixed center of mass.

A point of cardinal interest in this lecture is the investigation of the rolling and the fixed cones (polhode and herpolode of the top motion), which, by Poinsot's theorem are adapted to describe all rotations about a fixed center. The object in quest here is an expression of the rotation about the instantaneous axis, or preferably of the component rotations about the three movable axes, X, Y, Z, fixedin the body in terms of Klein's parameters α, β, γ, δ; *i. e.*, virtually to refer the rotation to the axes x, y, z, fixed in space. The results again show the remarkable adaptation of the new parameters to the problem in hand. When the three principal moments of inertia are equal, both polhode and herpolode turn out to be elliptic plane curves of the first degree. Thus both polhode and herpolode of the top's motion would be polhodes of two corresponding Poinsot motions; recalling the theorem of Jacobi that the motion of a top may be expressed as the relative divergence of two Poinsot motions.

Finally the motion of the polepoint, already briefly sketched for motion in real time, is resumed, in relation to complex time, to fully bring out the power of the elliptic functions α, β, γ, δ. Attention is first given to the parallelogram of periods in the t plane, in order to show the limits traced by the pole point on the ζ sphere. Indeed, the investigation is advantageously thrust back a step further by considering ζ as the image of the corresponding four half sheets of the $\sqrt{U}$ surface. It is hardly possible to follow Klein through this involved discussion here without reproducing his figures and computation in full. Suffice it to say that the stereographic projection of the ζ image from the top ($z = r$, $\zeta = \infty$) of the z axis, on the xy plane, is mapped out in correspondence with the parallelogram of periods on the plane of complex time, or for each point of the two positive and negative half sheets of the $\sqrt{U}$ surface.

The lecture concludes with a demonstration showing that a free body in hyperbolic non Euclidian space may be so fashioned as in real time to carry out the actual motions of the top. The form of such a body and the forces which actuate it are specified. Klein lays great stress on the beauty of this generalization.

In the fourth lecture, as already intimated, the top is set spinning on a horizontal plane with its point of support free to roam at pleasure, so that the top now has 5 degrees of freedom. In any case, however, the horizontal motion of the center of mass is uniform, and this point may, therefore, without essential restriction be considered fixed. But if the origin of coordinates be taken at the center of mass the problem returns to 3 degrees of freedom, with the difference that a new term equivalent to its vertical motion must make its appearance in the expression for kinetic energy. Hence a new treatment of the equations of motion is necessary, and if Eulerian coordinates be again introduced the method sketched in the 2d lecture is applicable throughout. The result for t, φ, ψ now, however, lead to hyperelliptic integrals, as for instance,

$$t = \int \frac{\sqrt{(1 + Ps) - Psu^2}}{\sqrt{U}} du$$

(where s is the distance between the centers of support and of mass and P the static moment), with a corresponding increase of the difficulty of the problem. The two new roots in the integrand thus make the corresponding Riemann surface two-leaved with six-branch points; but Klein shows that the parameters $\alpha, \beta, \gamma, \delta$ are again singularly adapted for the treatment of the present case, with this fatal difference, that for a single point in the t plane there correspond an infinite number of value of u. Hence as u is no longer a single valued function of t, it becomes necessary to seek a new function of which complex t, α, β, γ, δ shall all be single valued dependents. Such functions are the automorphic functions (η) obtained from elliptic functions by generalizing their periodicity. The line of argument above can now be broadened; construct in the η plane a rectangular hexagon which is the image of a half-sheet of the Riemann surface on the u plane, and which on reproduction covers the plane of complex η conformally and simply. Then to each point on the η plane there corresponds a single point on the Riemann surface; or

$$u, \quad \sqrt{U}, \quad \sqrt{1 + Ps - Psu^2}, \quad \alpha, \beta, \gamma, \delta,$$

are all single valued functions of η. Thus η quite replaces the t in the special case, and Klein carries out his analogies in detail by expressing the automorphic functions in terms of quotients of what he calls prime forms. Hence α, β, γ, δ are now given in terms of quotient of simple prime forms of η-functions, while they were above given as quotients of simple σ-functions. The full geometry of the case is not carried out in these lectures, however, and Klein regrets that the development of the automorphic functions has recently fallen into abeyance.

The reviewer is aware that with all endeavor he has given but an imperfect account of this remarkable book. That Klein's researches constitute a splendid advance in the dynamics of the rotation of a rigid body there can be no question. One cannot but hope that the outline given in these Princeton lectures may soon be expanded and put in shape more easily assimilable by persons moderately versed in the theory of elliptic functions. The boon of an appropriate lemma is ideal generosity, and not even a mathematician can scorn its almost mathematical elegance. A man may be a thoroughgoing soldier enough on land; but put him in the foot ropes of the flying jibboom in a storm, and he is apt to cut a most ludicrous figure. Shift a physicist's foothold of Cartesian differential coefficients, suspend him over an abyss of non-Euclidian space, and he will kick sturdily. Poor policy this, for a missionary!

CARL BARUS.

BROWN UNIVERSITY, PROVIDENCE, R. I.

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*THE TRANSMISSION OF RADIANT HEAT BY GASES AT VARYING PRESSURES.**

BEFORE describing my own investigations on the transmission of heat by gases, I shall refer briefly to the classical work of a somewhat similar nature by MM. Dulong and Petit early in the present century, 'Researches on the Measure of Temperatures, and on the Laws of Communication of Heat,' *Ann. of Phil.*, 1819.

In their researches on the 'Communication of Heat,' Dulong and Petit used as the cooling body a very large thermometer bulb filled with mercury, and as the recipient of the heat a large copper bulb or 'Balloon' about three decimeters in diameter, in the center of which the thermometer bulb was placed. The copper balloon was coated with lamp-black on the inside, and kept at any desired constant temperature by means of a water-bath or melting ice. The thermometer tube was of such length as to bring the zero of the scale outside the balloon; and the thermometer was adapted to be removed, heated and quickly replaced, air-tight. The balloon was connected with an air-pump capable of rapidly exhausting it down to about two millimeters pressure, and also with a gas-holder from which it could be quickly filled with the gas whose cooling properties were to be determined. The rate or 'Velocity' of cooling of the thermometer bulb was deduced from observations of the falling temperature at equal intervals of time.

With this apparatus Dulong and Petit made many carefully conducted experiments at differences of temperature between the thermometer and balloon ranging as high as 300 degrees; and with several different gases besides air, ranging in pressure from atmospheric to two millimeters. From the results of these experiments they deduced several laws of cooling which they held to be general in their application. They sharply divided the cooling into two parts: that due to convection—the actual contact of the surrounding cooler gas renewed by its own currents, and that due purely to radiation—the same as would occur in an 'absolute vacuum.' They derived a constant value for the latter, and values for the former varying with different gases and different pressures. They generally used the thermometer bulb naked, with its natural vitreous surface, but sometimes they silvered it. While this radical change in the character of surface greatly changed the loss of heat due to radiation, it apparently had no effect on that due to convection.

MM. Dulong and Petit fell into the grave error of deducing the behavior of the last

* Abstract of a paper read before the American Association for the Advancement of Science, August 10, 1897.