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CONTENTS:

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AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

*A CHAPTER IN THE HISTORY OF MATHEMATICS.**

ON the 10th of March, 1897, a hundred years after its original presentation, the Royal Academy of Sciences and Letters of Denmark published a French translation of a memoir by Caspar Wessel, entitled *Om Direktionens analytiske Betegning, et Forsög, anvendt fornemmelig til plane og sphaeriske Polygoners Oplösning*, or an Essay on the Analytic Representation of Direction, with Applications in Particular to the Determination of Plane and Spherical Polygons.

This paper, which deals with the geometric representation of imaginary quantities; which was read and printed several years before the famous essay of Argand and contains fully as exact a treatment of the subject, lay buried for nearly a century until attention was again drawn to it in 1895 by a thesis of S. D. Christensen upon the development of mathematics in Denmark and Norway in the eighteenth century.

Inasmuch as this memoir of Wessel's is still comparatively unknown, I have thought that it would not be uninteresting at this time to present a sketch of the development of the geometric treatment of the imaginary, particularly in the latter part of the eighteenth century and the first part of the nineteenth.

We find the square root of a negative quantity appearing for the first time in the *Stereometria* of Heron of Alexandria, 100 B. C. After having given a correct formula for the determination of the volume of a frustum of a pyramid with square base and applied it successfully to the case where the side of

*Address by the Vice-President before Section A, Mathematics and Astronomy.

the lower base is 10, of the upper 2, and the edge 9, the author endeavors to solve the problem when the side of the lower base is 28, of the upper 4, and the edge 15. Instead of the square root of 81–144 required by the formula, he takes the square root of 144–81 and calls it equal to 8 less $\frac{1}{16}$, *i. e.*, he replaces $\sqrt{-1}$ by 1, and fails to observe that the problem as stated is impossible. Whether this mistake was due to Heron or to the ignorance of some copyist cannot be determined.

In the solution of the problem to find a right angled triangle whose perimeter is 12 and area 7, Diophantus, in his *Arithmetica*, 300 A. D., reaches the equation $336x^2+24=172x$ and says that the equation cannot be solved unless the square of the half coefficient of x diminished by the product of 24 and the coefficient of x^2 is a square. No notice is taken of the fact that the value of x in this equation actually involves the square root of a negative quantity.

Bhaskara, born 1114 A. D., in his chapter *Vija Ganita*, was able to go a step further. He gave the rule:

> The square of a positive number as also of a negative number is positive and the square root of a positive number is twofold, positive and negative. There is no square root of a negative number, for this is not a square.

The first mathematician who had the courage actually to use the square root of a negative number in computation was Cardano. At an earlier period he had declared such a quantity to be wholly impossible, but in the *Ars Magna*, 1545, he discusses the problem of dividing 10 into two parts whose product shall be 40 and obtains the values $5+\sqrt{-15}$, $5-\sqrt{-15}$. These he verifies by multiplication. Such quantities he calls sophistic, since it is not permissible to operate with them as with pure negative numbers or others, nor to assign them a meaning.

Bombelli, in his *Algebra*, 1572, gives a number of rules for the use of such quantities as $a+b\sqrt{-1}$, but makes no endeavor to explain their character.

Girard knew that every equation has as many roots as its degree indicates and consequently recognized the existence of imaginary roots. In his *Invention nouvelle en l' algèbre*, 1629, while discussing the roots of the equation $x^4-4x+3=0$ he asks what purpose is subserved by such roots as $-1+\sqrt{-2}$ and $-1-\sqrt{-2}$ and says that they show the generality of the law of formation of the coefficients and are useful of themselves.

Descartes, in his *Geometria*, 1637, gives us no new ideas upon the subject, but is the first to apply the terms real and imaginary by way of contrast to the roots of an equation.

Wallis, in his *Treatise of Algebra*, 1685, leads the van in his endeavor to give a geometric interpretation to the square root of a negative number. In chapter LXVI we read:

> These *Imaginary* Quantities (as they are commonly called) arising from the *Supposed* Root of a Negative Square (when they happen,) are reputed to imply that the Case proposed is Impossible.
>
> And so indeed it is, as to the first and strict notion of what is proposed. For it is not possible that any Number (Negative or Affirmative) Multiplied into itself can produce (for instance) —4. Since that Like Signs (whether + or —) will produce +; and therefore not —4.
>
> But it is also Impossible that any Quantity (though not a Supposed Square) can be *Negative*. Since that it is not possible that any *Magnitude can be Less than Nothing* or any *Number Fewer than None*.
>
> Yet is not that Supposition (of Negative Quantities,) either Unuseful or Absurd; when rightly understood. And though, as to the bare Algebraick Notation, it import a Quantity less than nothing. Yet, when it comes to a Physical Application, it denotes as Real a Quantity as if the Sign were +; but to be interpreted in a contrary sense.

He illustrates this by distances measured forward and backward upon a straight line in the usual way, and continues:

> Now what is admitted in Lines must, on the same Reason, be allowed in Plains also.

Having thus justified the existence of negative planes, he goes on:

But now (supposing this Negative Plain, —1600 Perches, to be in the form of a Square;) must not this Supposed Square be supposed to have a Side? And if so, what shall this Side be?

We cannot say it is 40, nor that it is -40 * *

But thus rather that it is $\sqrt{-1600}$, or * * $10\sqrt{-16}$, or $20\sqrt{-4}$, or $40\sqrt{-1}$.

Where $\sqrt{}$ implies a Mean Proportional between a Positive and a Negative Quantity. For like as $\sqrt{bc}$ signifies a Mean Proportional between $+b$ and $+c$; or between $-b$ and $-c$; * * So doth $\sqrt{-bc}$ signify a Mean Proportional between $+b$ and $-c$, or between $-b$ and $+c$.

In chapter LXVII Wallis gives a geometric exemplification of a mean proportional, interpreting $\sqrt{bc}$ as a sine in a circle whose diameter $=b+c$, and $\sqrt{-bc}$ as a tangent in a circle whose diameter $=-b+c$. He then finds the base of a triangle when the two sides and the angle opposite, and hence the altitude, are given. Assuming $AP=20$, $PB=15$, and the altitude $PC=12$, by the use of the triangle BCP, right-angled at C, he obtains two values for the base AB. Then taking $AP=20$, $PB=12$, and the altitude $PC=15$, he finds imaginary values for the base.

These he interprets by saying:

This Impossibility in *Algebra* argues an Impossibility of the case proposed in Geometry; and that the Point B cannot be had, (as supposed,) in the Line AC, however produced (forward or backward,) from A.

Yet there are Two Points designed (out of that Line, but) in the same Plain; to either of which, if we draw the Lines AB, BP, we have a Triangle; whose Sides, AP, PB, are such as were required: And the Angle PAC, and Altitude PC, (above AC, though not above AB,) such as was proposed:

In this case he takes the triangle BCP to be right angled at B. Further:

And (in the Figure,) though not the Two Lines themselves, AB, AB, (as in the First case, where they lay in the Line AC:) yet the Ground-Lines on which they stand, $A\beta$, $A\beta$, are equal to the Double of AC: That is, if to either of those AB, we join Ba, equal to the other of them, and with the same Declivity; ACa (the distance of Aa) will be a Straight Line equal to the double of AC; as is ACa in the First case.

The greatest difference is this; that in the first Case, the Points B, B, lying in the Line AC, the Lines AB, AB, are the same with their Ground-Lines, but not so in this last case where B, B are so raised above $\beta\,\beta$ (the respective Points in their Ground-Lines, over which they stand), as to make the case feasible; (that is, so much as is the versed sine of CB to the Diameter PC:) But in both ACa (the Ground-Line of ABa) is equal to the Double of AC.

So that, whereas in case of Negative Roots, we are to say, The Point B cannot be found, so as is supposed in AC Forward, but Backward from A it may in the same Line: We must here say, in case of a Negative Square, the Point B cannot be found so as was supposed, in the Line AC; but Above that Line it may in the same Plain. This I have the more largely insisted upon, because the Notion (I think) is new; and this, the plainest Declaration that at present I can think of, to explicate what we commonly call the *Imaginary Roots* of Quadratic Equations. For such are these.

From these extracts it is evident that Wallis possessed, at least in germ, some elements of the modern methods of addition and subtraction of directed lines.

For the next hundred years no advance of importance was made. Euler, for example, makes large use of the imaginary, but in his *Algebra*, 1770, he observes:

All such expressions as $\sqrt{-1}$, $\sqrt{-2}$, etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.

On the 10th of March, 1797, a surveyor named Wessel presented to the Royal Academy of Sciences and Letters of Denmark a memoir 'On the Analytic Representation of Direction,' which was printed in 1798 and appeared in Vol. V, of the Memoirs of the Academy in 1799.

Caspar Wessel was born June 8, 1745, at Jonsrud, in Norway, where his father was a pastor. Though one of thirteen children, he had a good education, for in 1757 he entered the high school at Christiania and in 1763 went to Copenhagen to pursue further studies. In 1764 he was engaged by the Academy of Sciences as an assistant in the triangulation and preparation of a map of Denmark. Till 1805 he remained in the continuous employ of the Academy as surveyor. Wessel was highly esteemed by his contemporaries, and for some special work done after leaving the service of the Academy he received the Academy's silver medal and a full set of its memoirs. In 1819, when many of its maps were declared out of date, the trigonometric determinations of Wessel were made a special exception. In 1778 he passed an examination in Roman law. In 1815 he was made a Knight of the Danebrog. He died in 1818.

While Wessel was always well spoken of as a surveyor, he was never mentioned as a mathematician. Still the fact that his paper was the first to be accepted by the Academy from one not a member argues in his favor. This acceptance was due to Tetens, Councillor of State, to whom the MS. had been shown and whose assistance in improving it was acknowledged. In the History of the Academy of Sciences of Denmark published in 1843 Professor Jürgensen classes Wessel with others in the statement, "The treatises of the other mathematicians are monographs of no considerable scientific value," or "They are too special to be discussed more at length."

In the introduction to his memoir Wessel says:

The present essay has for its object to determine how to express segments of straight lines when we wish by means of a unique equation between a single unknown segment and other given segments to find an expression representing at once the length and direction of the unknown segment.

To be able to answer this question I shall employ two considerations which seem to me evident. In the first place, the variation of direction which may be produced by algebraic operations ought also to be represented by their symbols. In the second place we submit direction to algebra only by making its variation depend upon algebraic operations. Now according to the ordinary conception we can transform it by these operations only into the opposite direction, that is, from positive into negative and reciprocally. It follows that these two directions only would be susceptible of an analytic representation adapted to the usual conception and that the solution of the problem would be impossible for other directions. It is probably for this reason that nobody has given attention to this subject. Doubtless nobody has felt at liberty to change the definition of these operations once adopted. To this there is no objection so long as the definition is applied to ordinary quantities; but there are special cases where the peculiar nature of the quantities seems to invite us to give particular definitions to the operations. Then if we find these definitions advantageous it seems to me that we ought not to reject them. For in passing from arithmetic to geometric analysis, that is to say, from operations relative to abstract numbers to operations upon segments of a straight line, we shall have to consider quantities which may have to one another not only the same relations as abstract numbers, but also a great number of new relations. Let us try then to generalize the signification of our operations; let us not restrict ourselves, as has been done hitherto, to the employment of segments of a straight line in the same or opposite senses, but extend a little the notion of the way in which they are applied not only to the same cases as heretofore, but to an infinite number of other cases. If at the same time that we take this liberty we have respect to the ordinary rules of operations we in no way contravene the ordinary theory of numbers, but we merely develop it, we accommodate ourselves to the nature of the quantities and observe the general rule which requires us to render a difficult theory little by little more easy to comprehend. It is not then absurd to demand that in geometry operations be taken in a broader sense than in arithmetic. We shall admit without difficulty that it will be possible to vary the direction of segments in an infinite number of ways. Precisely by this means (as we shall show later) we succeed not only in avoiding all impossible operations and in explaining the paradox that it is necessary sometimes to resort to the impossible to obtain the possible, but we also succeed in expressing the direction of line-segments situated in the same plane quite as analytically as their length, without the memoir being embarrassed by new symbols or new rules. Now it must be agreed that the general demonstration of geometric theorems often becomes easier when we express direction in an analytic manner and submit it to the rules of algebraic operations than when we are compelled to represent it by figures which are applicable only to particular cases.

For these reasons I have proposed to myself:

1° to give the rules of operations of this nature;

2° to show by examples the application to cases where the segments are found in the same plane;

3° to determine by a new method not algebraic the direction of segments situated in different planes;

4° to deduce the general solution of plane and spherical polygons;

5° to deduce in the same way the known formulæ of spherical trignometry.

This, in brief, is an outline of the present memoir. I was led to write it by my desire to find a method which would enable me to avoid impossible operations; having discovered it I have made use of it to convince myself of the generality of certain known formulæ.

How well the author succeeds in carrying out his plan is shown by the memoir itself. Wessel says:

The addition of two segments is effected in the following manner: we combine them by drawing the one from the point where the other terminates; then we join by a new segment the two ends of the broken line thus determined.

He extends the definition to more than two segments and affirms:

In the addition of segments, the order of terms is arbitrary and the sum always remains the same.

His definition of the product of two segments is especially noteworthy:

The product of the two line-segments ought in every respect to be formed with one of the factors in the same way as the other factor is formed, with the positive or absolute segment taken equal to unity; that is to say:

1° The factors ought to have such a direction that they can be placed in the same plane as the positive unit;

2° As to length the product should be to one of the factors as the other is to the unit;

3° As to the direction of the product, if we draw from the same origin the positive unit, the factors and the product, the latter ought to be in the plane of the unit and the factors, and ought to deviate from one of the factors by as many degrees and in the same sense as the other deviates from the unit so that the angle of direction of the product or its deviation with respect to the positive unit is equal to the sum of the angles of direction of the factors.

Let us designate by $+1$ the positive rectilinear unit, by $+\varepsilon$ another unit perpendicular to the first and having the same origin; then the angle of direction of $+1$ will be equal to 0°, that of -1 to 180°, that of $+\varepsilon$ to 90° and that of $-\varepsilon$ to $-90°$ or to 270°; and according to the rule that the angle of direction of the product is equal to the sum of the angles of the factors, we shall have: $(+1).(+1)=+1$, $(+1).(-1)=-1$, $(-1)\cdot(-1)=+1$, $(+1)\cdot(-\varepsilon)=-\varepsilon$, $(-1)\cdot(+\varepsilon)=-\varepsilon$, $(-1)\cdot(-\varepsilon)=+\varepsilon$, $(+\varepsilon)\cdot(+\varepsilon)=-1$, $(+\varepsilon)\cdot(-\varepsilon)=+1$, $(-\varepsilon)\cdot(-\varepsilon)=-1$. Hence it follows that ε is equal to $\sqrt{-1}$ and that the deviation of the product is determined so that we violate none of the ordinary rules of operation.

It is interesting to note that while Wessel makes the addition and multiplication of directed lines a matter of definition, Argand, in his famous memoir of 1806, *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*, says: "Inasmuch as these principles depend upon inductions which are not securely established, they cannot as yet be considered as other than hypotheses whose acceptance or rejection should depend upon either the consequences which they entail or a more rigorous logic," although in his last contribution to the *Annales de Gergonne* he grants that this difficulty will vanish if with M. Français we define what is meant by a ratio of magnitude and position between two lines.

After explaining that if v represents any angle, and $\sin v$ a segment equal in length to the sine, positive when the measuring arc terminates in the first semicircumference and negative when it terminates in the second, $\varepsilon \sin v$ will express the sine of the angle v in direction and magnitude, Wessel shows that any radius making the angle v with the positive unit will equal $\cos v+\varepsilon \sin v$. In the multiplication of two radii $\cos v+\varepsilon \sin v$, $\cos u+\varepsilon \sin u$, he establishes the distributive law by reference to the formulæ,

$$\sin (v+u) = \sin v \cos u + \cos v \sin u,$$
$$\cos (v+u) = \cos v \cos u - \sin v \sin u,$$

in contrast to Argand, who assumes the distributive law and then derives the trigonometric formulæ.

A statement in this connection is noteworthy:

But if we have to multiply line segments which are not both in the plane passing through the absolute unit we cannot apply the preceding rule. For this reason I do not consider the multiplication of such segments.

The treatment of division follows in a natural manner, and it is proved that indirect quantities share with direct quantities the property that if the dividend is a sum we obtain by dividing each term of the sum by the divisor several quotients whose sum is the quotient sought.

Then comes a discussion of powers and roots establishing the fact that $(\cos v + \varepsilon \sin v)^{\frac{1}{m}}$ has m different values and only m. In the next paragraph Wessel shows that the m^{th} power of a line-segment may be put in the form $e^{ma+mb\sqrt{-1}}$, where e^{ma} represents the length and mb the angle of direction, and that thus we have a new method of representing the direction of line-segments in the same plane by the aid of natural logarithms. This last is not again referred to, but it is readily seen that Wessel was in possession of all three of the present methods of representing the complex number,

$a+b\sqrt{-1}$, $r(\cos\varphi+\sqrt{-1}\sin\varphi)$ and $re^{\phi\sqrt{-1}}$.

At the close of this section the author remarks:

At another time, with the permission of the Academy, I will present the complete proofs of these theorems. Having given an account of the way in which we must, in my judgment, understand the sum, the product, the quotient and power of line segments, I shall restrict myself to a few applications of the method.

The first application is to a demonstration of Cotes's theorem in which the fundamental theorem of algebraic equations is assumed as previously established. The second is to the resolution of plane polygons. In this certain characteristic notations occur. The first side of the quadrilateral considered is taken equal to the absolute unit; the sides in order beginning with the first are designated by the even numbers II, IV, VI, VIII, while I, III, V, VII, represent their deviations (in degrees) each with respect to the preceding side prolonged, regarding these deviations as positive or negative according as they have the same sense as the diurnal motion of the sun or the opposite; I′, III′, V′, VII′ denote the expressions $\cos \mathrm{I}+\varepsilon \sin \mathrm{I}$, etc., while $\mathrm{I}^{-\prime}$, $\mathrm{III}^{-\prime}$, $\mathrm{V}^{-\prime}$, $\mathrm{VII}^{-\prime}$ denote the expressions $\cos(-\mathrm{I})+\varepsilon\sin(-\mathrm{I})$ or $\cos \mathrm{I}-\varepsilon \sin \mathrm{I}$, etc.

The author then deduces the two formulæ,

$$\mathrm{II} + \mathrm{IV}\cdot\mathrm{III}' + \mathrm{VI}\cdot\mathrm{III}'\cdot\mathrm{V}' + \mathrm{VIII}\cdot\mathrm{III}'\cdot\mathrm{V}'\cdot\mathrm{VII}' = 0,$$

$$\mathrm{II}\cdot\mathrm{III}'\cdot\mathrm{V}'\cdot\mathrm{VII}' + \mathrm{IV}\cdot\mathrm{V}'\cdot\mathrm{VII}' + \mathrm{VI}\cdot\mathrm{VII}' + \mathrm{VIII} = 0,$$

and proves that two equations of this form will suffice for the solution of any polygon in which the only unknown parts are three angles, or two angles and a side, or an angle and two sides.

Wessel next attacks the problem of representing the direction of any line segment in space by taking it as the radius, r, of a sphere. Assuming three perpendicular radii as axes and denoting positive unit lengths upon these, to the left by 1, forward by ε and upward by η respectively, where $\varepsilon^2 = -1$, and $\eta^2 = -1$, he concludes that a radius whose extremity has for coordinates $x, \eta y, \varepsilon z$ will be properly designated by $x + \eta y + \varepsilon z$. Defining the plane of r and εr as the horizontal plane and that of r and ηr as the vertical plane, he examines the effect of moving the extremity through an arc of I degrees parallel to the horizontal plane and obtains for $x + \eta y + \varepsilon z$ the new value,

$$\eta y + (x + \varepsilon z)(\cos \mathrm{I} + \varepsilon \sin \mathrm{I}) = \eta y + x\cos \mathrm{I} - z \sin \mathrm{I} + \varepsilon x \sin \mathrm{I} + \varepsilon z \cos \mathrm{I},$$

in which the term ηy remains unchanged. This operation he indicates by the use of the sign ,, as $(x + \eta y + \varepsilon z)$,, (cos I + ε sin I) and says that it has only imperfectly the signification of a sign of multiplication, for the operation leaves unchanged that one of the segments occurring in the multiplicand which is outside of the plane corresponding to the rotation indicated by the multiplier. He calls attention to the fact that the factors must be used in order from left to right. Similarly when the extremity of the radius moves through an arc of II degrees parallel to the vertical plane we have

$$(x + \eta y + \varepsilon z) \;,,\; (\cos \mathrm{II} + \eta \sin \mathrm{II}) = \varepsilon z + x \cos \mathrm{II} - y \sin \mathrm{II} + \eta x \sin \mathrm{II} + \eta y \cos \mathrm{II}.$$

It follows at once that

$$(x + \eta y + \varepsilon z) \;,,\; (\cos \mathrm{I} + \varepsilon \sin \mathrm{I}) \;,,\; (\cos \mathrm{III} + \varepsilon \sin \mathrm{III}) = (x + \eta y + \varepsilon z) \;,,\; (\cos (\mathrm{I} + \mathrm{III}) + \varepsilon \sin (\mathrm{I} + \mathrm{III}))$$

and

$$(x + \eta y + \varepsilon z) \;,,\; (\cos \mathrm{II} + \eta \sin \mathrm{II}) \;,,\; (\cos \mathrm{IV} + \varepsilon \sin \mathrm{IV}) = (x + \eta y + \varepsilon z) \;,,\; (\cos (\mathrm{II} + \mathrm{IV}) + \varepsilon \sin (\mathrm{II} + \mathrm{IV}))$$

also that

$$x + \eta y + \varepsilon z = (x + \eta y + \varepsilon z) \;,,\; (\cos I + \varepsilon \sin \mathrm{I}) \;,,\; (\cos \mathrm{I} - \varepsilon \sin \mathrm{I}) = (x + \eta y + \varepsilon z) \;,,\; (\cos \mathrm{II} + \eta \sin \mathrm{II}) \;,,\; (\cos \mathrm{II} - \eta \sin \mathrm{II}).$$

Wessel then studies the effect of alternate horizontal and vertical rotations. Representing the radius in its first position by s and in its final position by S, and denoting the arcs in order by I, II, III, * * * *VI*, he obtains the formula

$$S = s \;,,\; \mathrm{I}' \;,,\; \mathrm{II}' \;,,\; \mathrm{III}' \;,,\; \mathrm{IV}' \;,,\; \mathrm{V}' \;,,\; \mathrm{VI}'.$$

In this connection he observes that such factors as V' ,, VI' can be transferred to the first member by using their reciprocals in inverse order, as

$$S \;,,\; \mathrm{VI}^{-\prime} \;,,\; \mathrm{V}^{-\prime} \;,,\; \mathrm{IV}^{-\prime} = s \;,,\; \mathrm{I}' \;,,\; \mathrm{II}' \;,,\; \mathrm{III}' \;,, .$$

These results are applied to the solution of spherical polygons and the determination of the properties of spherical triangles. As in the case of plane polygons, I, II, III, etc., represent the exterior angles and sides in order, the odd numbers the angles, and the even numbers the sides. Supposing the angles and the sides of a polygon known except one angle and two sides, or two angles and a side, or three angles, or three sides, the unknown parts can be determined by the equation

$$s \;,,\; \mathrm{I}' \;,,\; \mathrm{II}' \;,,\; \mathrm{III}' \;,,\; \mathrm{IV}' \;,,\; \mathrm{V}' \;,,\; \mathrm{VI}' \;,,\; \cdots \;,,\; N' = s,$$

where s is indeterminate, and may be supposed equal to r, εr, or ηr. The effect of the rotations indicated by this equation is to submit the sphere alternately to rotations about the axis of the horizon and the axis of the vertical circle so that each point of the sphere describes first a horizontal arc which measures the first exterior angle of the polygon, then a vertical arc containing as many degrees as the first side of the polygon, then a new horizontal arc which measures the second angle, etc. The sphere finally returns to its original position, while each of its points has described as many horizontal arcs as the polygon has angles and as many vertical arcs as it has sides.

While Wessel's results, as obtained by these alternate rotations, are correct so far as they go, he fails to observe that a general rotation must be compounded of three rotations about the axes ε, η, ε or η, ε, η. Stranger still he makes no study of rotations about the real axis. Thiele, in his introduction to Wessel's memoir, shows how easy it would have been to go a few steps further and arrive at the notion of quaternions. But be that as it may, Wessel deserves great credit for having devised the only successful method of dealing with line-segments in space previous to the work of Hamilton beginning in 1843.

Unmindful of Euler's demonstration of the real value of $(\sqrt{-1})^{\sqrt{-1}}$ Argand endeavors to show that such an expression may be used to represent a directed line in space. Français tries to solve the problem by the use of imaginary angles, but frankly acknowledges his failure. Servois sees with remarkable clearness what is needed, but is unable to reach it. He says:

The table of double argument which you (Gergonne) propose, as applied to a plane supposed to be so divided into points or *infinitesimal* squares that each square corresponds to a number which would be its *index*, would very properly indicate the length and position of the radii vectores which revolve about the point or central square corresponding to ± 0; and it is quite remarkable that if we designated the length of a radius vector by a, and the angle it makes with the real line......, -1, ± 0, $+1$...... by a, the rectangular coordinates of its *extremity remote from the origin* by x, y, the real line being the axis of x, the point would be determined by $x+y\sqrt{-1}$...... It is clear that your ingenious tabular arrangement of numerical magnitudes may be regarded as a central slice (*tranche centrale*) of a table of triple argument representing points and lines in tri-dimensional space. You will doubtless give to each term a trinomial form; but what would be the coefficient of the third term? For my part I cannot tell. Analogy would seem to indicate that the trinomial should be of the form $p\cos\alpha+q\cos\beta+r\cos\gamma$, α, β, and γ being the angles made by a right line with three rectangular axes and that we should have

$$(p\cos\alpha+q\cos\beta+r\cos\gamma)\,(p'\cos\alpha+q'\cos\beta+r'\cos\gamma)=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

The values of p, q, r, p', q', r' satisfying this condition would be *absurd*, but would they be imaginaries, reducible to the general form $A+B\sqrt{-1}$?

As we all know now, these non-reals which Servois could not determine may be identified with the $+i$, $+j$, $+k$, $-i$, $-j$, $-k$, of Hamilton's Quaternions.

In 1799, in his first published paper, *Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse*, the celebrated Gauss, then only twenty-two years of age, says:

By an imaginary quantity I always understand here a quantity contained in the form $a+b\sqrt{-1}$, so long as b is not zero. * * * If imaginary quantities are to be retained in analysis (which for many reasons seems better than to abolish them, provided they are established on a sufficiently solid foundation) it is necessary that they be considered as equally possible with real quantities, on which account I should prefer to include both real and imaginary quantities under the common designation *possible quantities.* * * * A vindication of these (*i. e.*, imaginary quantities), as well as a more fruitful exposition of the whole matter, I reserve for another occasion.

This occasion, however, does not seem to have come till more than thirty years later. In the *Göttingische gelehrte Anzeigen* of April 23, 1831, in an account by Gauss of his own paper *Theoria residuorum biquadraticorum, Commentatio secunda*, we read:

Our general arithmetic, so far surpassing in extent the geometry of the ancients, is entirely the creation of modern times. Starting originally from the notion of absolute integers, it has gradually enlarged its domain. To integers have been added fractions, to rational quantities the irrational, to positive the negative and to the real the imaginary. This advance, however, has always been made at first with timorous and hesitating step. The early algebraists called the negative roots of equations false roots, and these are indeed so when the problem to which they relate has been stated in such a form that the character of the quantity sought allows of no opposite. But just as in general arithmetic no one would hesitate to admit fractions, although there are so many countable things where a fraction has no meaning, so we ought not to deny to negative numbers the rights accorded to positive simply because innumerable things allow no opposite. The reality of negative numbers is sufficiently justified since in innumerable other cases they find an adequate substratum. This has long been admitted, but the imaginary quantities—formerly and occasionally now, though improperly, called impossible—as opposed to real quantities are still rather tolerated than fully naturalized, and appear more like an empty play upon symbols to which a thinkable substratum is denied unhesitatingly by those who would not depreciate the rich contribution which this play upon symbols has made to the treasure of the relations of real quantities.

The author has for many years considered this highly important part of mathematics from a different point of view, where just as objective an existence may be assigned to imaginary as to negative quantities, but hitherto he has lacked opportunity to publish these views, though careful readers may find traces of them in the memoir upon equations which appeared in 1799 and again in the prize memoir upon the transformation of surfaces. In the present paper the outlines are given briefly; they consist of the following:

Positive and negative numbers can only find an application when the thing counted has an opposite which when conceived of as united with it has the effect of destroying it. Accurately speaking, this supposition can only be made where the things enumerated are not substances (objects thinkable in themselves), but relations between any two objects. It is postulated that these objects are arranged after a definite fashion in a series, *e. g.*, A, B, C, D, * * * and that the relation of A to B can be regarded as equal to that of B to C, etc. The notion of opposition involves nothing further than the *interchange* of the terms of the relation so that if the relation of (or transition from) A to B is considered as $+1$ the relation of B to A must be represented by -1. So far then as such a series is unlimited on both sides, every real integer represents the relation of a term arbitrarily taken as origin to a definite term of the series.

If, however, the objects are of such a kind that they cannot be arranged in one series, even though unlimited, but only in series of series, or, what amounts to the same thing, they form a manifoldness of two dimensions; if there is the same connection between the relations of one series to another, or the transitions from one to another, as in the case of the transition from one term of a series to another term of the same series, we shall evidently need for the measurement of the transition from one term of the system to another, besides the previous units $+1$ and -1, two others opposite in character $+i$ and $-i$. Obviously we must also postulate that the unit i shall always mark the transition from a given term of the one series to a definite term of the immediately adjacent series. In this way the system can be arranged in a two-fold manner in series of series.

The mathematician leaves entirely out of consideration the nature of the objects and the content of their relations. He has simply to do with the enumeration and comparison of the relations. So far as he has assigned sameness of nature to the relations designated by $+1$ and -1, considered in themselves, he is warranted in extending such sameness to all four elements $+1$, -1, $+i$, $-i$.

These relations can be made intuitive only by a representation in space and the simplest case, where there is no reason for arranging the objects in any other than quadratic fashion, is that in which an unlimited plane is divided into squares by two systems of parallel lines intersecting at right angles, and the points of intersection are selected as the symbols. Every such point has four adjacent points, and if we designate the relation A to a neighboring point by $+1$, the relation to be denoted by -1 is determined of itself, while we can select which of the two others we please for $+i$, or can take the point to be denoted by $+i$ at pleasure on the *right* or *left*. This distinction between right or left so soon as we have fixed (at pleasure) upon forwards and backwards *in* the plane, and above and below with respect to the two sides of the plane is completely determined *in itself*, although we can convey our own intuition of this difference to others *only* by reference to actually existent material things. But when we have decided upon the latter we see that it is still a matter of choice as to which of the two series intersecting at one point we shall regard as the principal series and which direction in it shall be considered as having to do with positive numbers. We see further that if we wish to take $+1$ for the relation previously expressed by $+i$, we must necessarily take $+i$ for the relation previously expressed by -1. In the language of mathematicians this means that $+i$ is *a* mean proportional between $+1$ and -1, or corresponds to the symbol $\sqrt{-1}$. We say purposely not *the* mean proportional because $-i$ has just as good a right to that designation. Here then the demonstrability of an intuitive signification of $\sqrt{-1}$ has been fully justified and nothing more is necessary to bring this quantity into the domain of objects of arithmetic.

We have thought to render the friends of mathematics a service by this brief exposition of the principal elements of a new theory of the so-called imaginary quantities. If people have considered this subject from a false point of view and thereby found a mysterious obscurity, this is largely due to an unsuitable nomenclature. If $+1$, -1, $\sqrt{-1}$ had not been called positive, negative, imaginary (or impossible) unity, but perhaps direct, inverse, lateral unity, such obscurity could hardly have been suggested. The subject which, properly enough, in the present treatise has been touched upon only incidentally the author has reserved for a more elaborate treatment in the future where also the question will be answered as to why the relations between things which present a manifoldness of more than two dimensions cannot furnish still other classes of magnitudes admissible in general arithmetic.

Such was Gauss's masterly presentation of the underlying principles of the treatment of the imaginary. In Germany the impulse given by his commanding influence is felt even to the present day.

Buée's memoir *Sur les Quantités Imaginaires*, read before the Royal Society of London in 1805 and covering 65 pages of the Philosophical Transactions of 1806, is somewhat vague and disappointing. He describes $\sqrt{-1}$ as follows:

$\sqrt{-1}$ is the sign of perpendicularity $\sqrt{-1}$ is not the sign of an arithmetical operation, nor of an arithmetico-geometric operation, but of an operation purely geometric. It is a purely descriptive sign which indicates the direction of a line without regard to its length.

Near the close of his paper he investigates what becomes of the conic sections when their coordinates become imaginary and decides that the circle passes into an equilateral hyperbola in the plane perpendicular to the plane of the circle and similarly for the other conics.

A further discussion of the justly celebrated epoch-making memoir of Argand and the contributions of himself, Français-Gergonne and Servois to the *Annales de Gergonne* from 1813 to 1815 is rendered the less necessary by reason of Houel's republication of all these papers in 1874 and their translation into English by Hardy in 1881.

It is interesting to note the early view of imaginaries entertained by so distinguished a mathematician as Cauchy. In his *Cours d' Analyse*, 1821, we read:

In analysis we apply the term symbolic expression or symbol to every combination of algebraic signs which signifies nothing by itself or to which we attribute a value different from that which it naturally ought to have. * * * * Among the symbolic expressions whose consideration is of importance in analysis we ought especially to distinguish those which are called imaginary. * * * * We write the formula

$$\cos(a+b)+\sqrt{-1}\sin(a+b)=$$
$$(\cos a+\sqrt{-1}\sin a)(\cos b+\sqrt{-1}\sin b).$$

The three expressions which the preceding equation contains * * * * are three symbolic expressions which cannot be interpreted according to generally established conventions and represent nothing real. * * * * The equation itself, strictly speaking, is inexact and has no meaning.

In 1849, however, in a paper *Sur les quantités géométriques*, in which he gives suitable credit to Argand, Français and others, he acknowledges:

In my *Analyse algébrique*, published in 1821, I was content to show that the theory of imaginary expressions and equations could be rendered rigorous by considering these expressions and equations symbolic. But after new and mature reflections the better side to take seems to be to abandon entirely the use of the sign $\sqrt{-1}$ and to replace the theory of imaginary expressions by the theory of quantities which I shall call geometric.

Having defined the term geometric quantity exactly as we now define the term vector and shown when two geometric quantities are equal, he continues:

The notion of *geometric quantity* will comprehend as a particular case the notion of *algebraic quantity*, positive or negative, and *a fortiori* the notion of *arithmetic quantity*. * * * We must further define the different functions of these quantities, especially their sums, their products and their integral powers by choosing such definitions as agree with those admitted when we are dealing with algebraic quantities alone. This condition will be fulfilled if we adopt the conventions now to be given.

Then follow the definitions called for, together with a treatment of the whole subject fully up to modern demands. Cauchy observes that a large part of the results of the investigations of Argand and others would seem to have been discovered as early as 1786 by Henri Dominique Truel, who communicated them about 1810 to Augustin Normand, of Havre.

In 1828 there appeared in Cambridge, England, a remarkable work by Rev. John Warren, entitled *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*. Though this book has latterly received scant credit, its merits were fully recognized by De Morgan and acknowledgments of indebtedness were frankly made by Hamilton.

Throughout Warren's work the term quantity, like Cauchy's geometric quantity, indicates a line given in length and direction. Some of his definitions are as follows:

The sum of two quantities is the diagonal of the parallelogram whose sides are the two quantities. The first of four quantities is said to have to the second the same ratio which the third has to the fourth; when the first has *in length* to the second the same ratio which the third has *in length* to the fourth, according to Euclid's definition; and also the angle at which the fourth is inclined to the third is equal to the angle at which the second is inclined to the first, and is measured in the same direction. Unity is a positive quantity arbitrarily assumed from a comparison with which the values of other quanties are determined. If there be three quantities such that unity is to the first as the second to the third, the third is called the *product*, which arises from the *multiplication* of the second by the first. If there be three quantities such that the first is to unity as the second is to the third, the first quantity is called the *quotient*, which arises from the *division* of the second by the third.

The fundamental laws of algebra as governing these quantities are established in their utmost generality with a rigor of reasoning that has probably not been surpassed. The author even goes so far as to deduce the binomial formula, to develop many series and to apply the methods of the differential and integral calculus to quantities of the class defined. In form Warren's work is intensely algebraic and fairly bristles with formulæ.

To sum up:

Caspar Wessel, in 1797, published the first clear, accurate and scientific treatment of directed lines in the same plane, as represented by quantities of the form $a + b\sqrt{-1}$, establishing the laws governing their addition, subtraction, multiplication and division, and showing these quantities to be of practical value in the demonstration of theorems and solution of problems; he also worked out a partial theory of rotations in space, so far as they can be decomposed into rotations about two axes at right angles.

Not very much later, 1799, Gauss indicated that he was in possession of a method of dealing with quantities of the form $a+b\sqrt{-1}$ which would consider them as equally possible with real quantities, but its fuller exposition was deferred till 1831.

Buée's paper of 1805 lays great emphasis upon $\sqrt{-1}$ as the sign of perpendicularity, but fails to give any satisfactory interpretation of the product of directed lines.

Argand's famous memoir of 1806 is hardly in danger of receiving too much credit. Though written after Wessel's paper there is not the slightest probability that Argand had any knowledge of the Norwegian surveyor, and, in fact, certain of his theorems are established less rigorously than by Wessel. Argand gave numerous applications of his theory to trigonometry, geometry and algebra, some of which are very noteworthy, especially his demonstrations of Ptolemy's theorem regarding the inscribed quadrilateral and of the fundamental proposition of the theory of equations.

The contributions of Français, Gergonne and Servois, 1813–1815, served to do away with some of the errors into which Argand had fallen and thus to give a clearer insight into the fundamental notions of the subject.

Though Warren's book of 1828 contains definitions differing but little from those of Wessel and Français and a notation which seems only a modification of that of Français, his generalized treatment of directed lines in the plane must be regarded as highly original.

Cauchy's work lay in the extension and development of the labors of his predecessors rather than in the introduction of new ideas.

Such were the beginnings of the study of the geometric representation of the imaginary which has led in modern times to the establishment of such great bodies of doctrine as the theory of functions on the one side and quaternions on the other, with the Ausdehnungslehre occupying a position between. Who can tell what the next century will bring forth?

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