Skating ponds illuminated by natural gas are among the possibilities of the future.

IRA REMSEN.

BALTIMORE, January 14, 1896.

'PROFESSORS' GARNER AND GATES.

THE daily papers state that Mr. Richard L. Garner, whose alleged investigation of the speech of monkeys has been so prominently advertised, is again expected in America. Accounts of the alleged investigations of Mr. Elmer Gates on the development of the brain are also being extensively reported. It is perhaps the duty of a scientific journal to state that neither of these gentlemen has as yet published scientific work deserving serious consideration. J. McK. C.

SCIENTIFIC LITERATURE.

The Psychology of Number and Its Applications to Methods of Teaching Arithmetic: By JAMES A. McLELLAN, A.M., LL.D., and JOHN DEWEY, Ph.D. International Educational Series. D. Appleton & Co., New York.

This book makes a false analysis of the number concept, but advocates methods in teaching arithmetic which are in the main good. The conviction of its authors that the difficulties which children have with arithmetic are due to the neglect of teachers to lay sufficient stress on the metrical function of number has carried them to the extreme of maintaining that number is essentially metrical in its nature and origin. The conviction is well founded, inasmuch as the first serious difficulties of children are with fractions whose primitive function was unquestionably metrical and to which men in general attach no other than a metrical meaning; but there is no reason for drawing the conclusion that because the fraction, which is but a secondary concept of arithmetic, is metrical, its primary concept, the integer, is metrical also, or even that because a child can hardly be made to understand fractions without associating them with measurement, he requires the same help with integers. Nevertheless, the authors of this book maintain, in the most unqualified manner, that the integer is essentially metrical and should be taught accordingly. Thus they account as follows for the origin of number: Man found himself in a world in which the supply of almost everything that he needed was limited. To obtain what he required, therefore, an economy of effort, a careful adjustment of means to an end, was necessary. But the process of adjusting means to an end is valuable in the degree in which it establishes an exact balance between them. "In the effort to attain such a balance, the vague quantitative ideas of smaller and greater * * * were transformed into the definite quantitative ideas of just so distant, so long * * *. This demands the introduction of the idea of number. Number is the definite measurement, the definite valuation of a quantity falling within a given limit."

They define counting, the fundamental numerical operation as but measuring with an undefined unit. "We are accustomed to distinguish counting from measuring. Nevertheless, all counting is measuring and all measuring counting. The difference is that in what is ordinarily termed counting, as distinct from measuring, we work with an undefined unit; it is vague measurement because our unit is unmeasured. * * * If I count off four books, 'book,' the unit which serves as unit of measurement, is only a *qualitative*, not a *quantitative* unit."

And they formally define number as 'the repetition of a certain magnitude used as the unit of measurement to equal or express the comparative value of a magnitude of the same kind,' a definition which, so far as it goes, agrees, it is true, with that given by Newton in his *Arithmetica Universalis*, viz, 'the abstract ratio of any quantity to another quantity of the same kind taken as unit,' though Newton's purpose having been to formulate a working definition comprehensive enough to include the irrational number, it is anything but evident that this statement represents his analysis of the notion of number in the primary sense.

The immediate objection to all this is that it is much too artificial to be sound. And in fact it requires but a little reflection to be convinced that pure number is not metrical and that counting is not measuring, but something so much simpler that men must have counted long before they knew how to measure in any proper sense.

It is not enough to say that counting is the simplest mathematical operation; it is one of the simplest of intellectual acts. For to count a group of things on the fingers is merely by assigning one of the fingers to each one of the things to form a group of fingers which stand in a relation of 'one-to-one correspondence' to the group of things. And counting with numeral words is not a whit more complex. The difference is only that words instead of fingers are attached to the things counted. But, the order of the words being invariable, the last one used in any act of counting is made to represent the result, for which it serves as well as the group of all that have been used would do. The group of fingers or this final numeral word answers as a register of the things by referring to which one may keep account of them as a child does of his marbles or pennies without remembering them individually, and this is the simplest and most immediate practical purpose that counting serves.

The number of things in any group of distinct things is simply that property of the group which the group of fingers—or, it may be, of marks or pebbles or numeral words—used in counting it represents, the one property which depends neither on the character of the things, their order nor their grouping, but solely on their distinctness. Gauss said with reason that arithmetic is the pure science *par excellence*. Even geometry and mechanics are mixed sciences in so far as their reality is conditioned by the correctness of the postulates they make regarding the external world. But the one postulate of arithmetic is that distinct things exist. It is an immediate consequence of this postulate that the result of counting a group of such things is the same whatever the arrangement or the character of the things, and this is the essence of the number-concept.

Counting, therefore, is not measuring and number is not ratio. Pure number does not belong among the metrical, but among the non-metrical mathematical concepts. The number of things in a group is not its measure, but, as Kronecker once said very happily, its 'invariant,' being for the group in relation to all transformations and substitutions what the discriminant of a quantic, say, is for the quantic in relation to linear transformations, unchangeable. Nor are the notions of numerical equality and greater and lesser inequality metrical. When we say of two groups of things that they are equal numerically, we simply mean that for each thing in the second there is one in the first and for each thing in the first there is one in the second, in other words that the groups may be brought into a relation of one-to-one correspondence, so that either one of them might be taken instead of a group of fingers to represent the other numerically. And when we say that a first group is greater numerically than a second, or that the second is less than the first, we mean that for each thing in the second there is one in the first, but not reciprocally one thing in the second for each in the first. Instead of comparing the groups directly we may count them separately on the fingers, and by a comparison of the results obtain the finger representation of the numerical excess of the one group over the other in case they are unequal. And this is all that is meant when we say that by counting we determine which of two groups is the larger and by how much.

It is therefore obvious, as for that matter our authors themselves urge, that the rational method of teaching a child the smaller numbers is by presenting to him their most complete symbols, corresponding groups of some one kind of thing as blocks, marbles or dots. By such aids he may be taught, with as great soundness as concreteness, not only the numbers themselves and their simple relations, but the meaning of addition, subtraction, multiplication and division of integers and the 'laws' which characterize these operations. This accomplished, he is ready to be taught notation and the addition and multiplication tables and to be practised on them until he has attained the art of quick and accurate reckoning. 'Measuring with undefined units' is a fiction with which there is no need to trouble him. For in however loose a sense the word may be used, 'measuring' at least involves the conscious use of a unit of reference. But no one ever did or ever will count a group of horses, for instance, by first conceiving of an artificial unit horse and then matching it with each actual horse in turn—which 'measuring' the group of horses must mean if it means anything. A conception of 'three' which makes 'three horses' mean in the last analysis 'three times a fictitious unit horse' does not differ so essentially as our authors think from the 'fixed unit' conception of this number against which they protest so strenuously. And this fictitious operation is no more the essence of multiplication and division than it is of counting. Multiplication of integers is abbreviated addition. The product 'three times two' is the sum of three two's not, happily, the measure in terms of a primary undefined unit of something whose measure in terms of a secondary undefined unit is three, when the measure of the secondary unit itself in terms of this primary unit is two.

On the other hand, measuring in the ordinary sense—the process which leads to the representation of *continuous* magnitudes as lines or surfaces, in terms of some unit of measure—deserves all the prominence which our authors would give it in arithmetic. We do not mean measuring in the exact mathematical sense, of course, but the rough measuring of common life, in which the magnitude measured and the unit are always assumed to be commensurable.

Compared with counting, or even addition and multiplication, an operation which involves the use of an arbitrary unit, and the comparison of magnitudes by its aid, is artificial. But this metrical use of number is of immense practical importance and of great interest to any child mature enough to understand it. No doubt a child may use a twelve-inch rule to advantage when practicing multiplication and division of integers. Certainly such an aid is almost indispensable in learning fractions. Without it the fraction is more than likely to be a mere symbol to him, without exact meaning of any kind. 'Two-thirds' has a reality for the child who can interpret it as the measure of a line two inches long in terms of a unit three inches long, which it quite lacks for him who can only repeat that it is 'two times the third part of unity.' Mathematicians now define the fraction as the symbolic result of a division which cannot be actually effected, but that definition will not serve the purposes of elementary instruction. It is as certain that the fraction had a metrical origin as it is that the integer had not, and in learning fractions, as in learning integers, the child cannot do better than follow the experience of the race.

Our authors must, therefore, be credited with doing the cause of rational instruction in arithmetic a real service by laying the stress they do on this proper metrical use of number. Their chapters on the practical teaching of arithmetic, moreover, though unduly prolix, contain many excellent suggestions. It is a pity that a book in the main so sound in respect to practice should be wrong on fundamental points of theory. One can but regret that its authors did not take pains before writing it to read what mathematicians of the present century have had to say on the questions with which they meant to deal. Their conception of number might have been modified by the considerations which have led mathematicians to 'arithmetise' the higher analysis itself by replacing the original metrical definition of the irrational number by a purely arithmetical one. At all events their notions of certain mathematical concepts would not have been so crude; they would not have made such a use of mathematical terms as this: "Quantity, the unity measured, whether a 'collection of objects' or a physical whole, is *continuous*, an undefined how *much;* number as measuring value is discrete, how *many.*"

H. B. FINE.

PRINCETON, December 31, 1895.

Experimental Farms. Reports for 1894. Printed by order of Parliament. Ottawa, 1895. 422 pp. 8°.

The direct application of scientific methods of investigation to practical questions has, perhaps, in no field found greater extension during the last decade on this continent than in agriculture.

The establishment of the experiment stations in connection with agricultural colleges in all our States by the Hatch Act of 1887 has revolutionized the possibilities of agricultural pursuits, and what this act did for the United States, Canada did the same year in perhaps a more efficient if not as extensive manner for its people. This greater efficiency we would attribute to the fact that the direction of the five experimental farms located in different parts of the country is concentrated in one director and one staff, thereby producing that unity of purpose which insures success.