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NON-EUCLIDEAN GEOMETRY.

BY G. A. MILLER, PH.D., EUREKA COLLEGE, EUREKA, ILL.

EUCLID'S elementary geometry was written about three centuries before the Christian era. We must conclude that it was much superior to all preceding works on this subject. Proclus, who wrote a commentary on Euclid's Elements in the fifth century of our era, represents it such, and his statements are corroborated by the facts that all similar works of Euclid's predecessors have ceased to exist, and, if any elementary geometry was written by a Greek after Euclid, there is no mention made of this anywhere.[1]

The facts that Euclid's Elements are still used as a text-book — especially in England — and that the works used in its place are generally based upon it, are perhaps still stronger evidences of its excellence.

No geometry can be written without making some assumptions with respect to the space with which it deals. These are generally of such a nature as to commend themselves to our full confidence by their mere mention, and are commonly called axioms. It is the duty of the geometer to demonstrate properties and relations of magnitudes by non-contradictory statements which rest ultimately upon these axioms. It is evident that the axioms should be as few and as clear as possible. Upon essentially different axioms essentially different geometries may be established.

Among the axioms of Euclid there is at least one which is not axiomatic.[2] This is the axiom of parallels, which reads as follows: —

"If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right-angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right-angles."

All the popular text-books on elementary geometry employ this axiom either in this form or in some shorter form, such as, "Through a point without a line only one line can be drawn parallel to the given line."

Many efforts have been made to demonstrate this axiom. Since it does not depend upon more elementary axioms, such attempts must be futile. If we assume it to be true, it follows directly that the sum of the three angles of a plane triangle is two right-angles; and, conversely, if we should assume that the sum of the internal angles of a plane triangle is two right-angles, this axiom would follow.[3]

As the geometers who do not adopt all the axioms of Euclid deny this, non-Euclidean geometry is sometimes defined as the geometry which does not assume that the sum of the three angles of a plane triangle is two right-angles. A more satisfactory definition is, non-Euclidean geometry is a geometry which assumes other properties of space in place of the following properties of Euclidean space: —

The sum of the three angles of a plane triangle is two right-angles, space is an infinite continuity of three dimensions, and rigid bodies may be moved in every way in space without change of form.

Just one hundred years ago (1792) the famous mathematician Gauss began the study of a geometry free from the first of these assumptions. He did not publish the results of his study. We may infer something in regard to them from his letters.[4] It was not until 1840 that a geometry was published in which Euclid's axiom of parallels was replaced by another, and the sum of the angles of a plane finite triangle was thus shown to be less than two right-angles. The work was written by a Russian mathematician named Lobatschewsky. It contains only sixty-one pages and bears the title "Geometrische Untersuchungen zur Theorie der Parallellinien." He began his treatment of parallels by observations, in substance, as follows: —

Given a fixed line (L) and a fixed point (A) not on this line. The lines through A lying in the plane determined by A and L may be divided with respect to L into two classes — (1) those intersecting L, and (2) those not intersecting L. The assumption that the second class consists of the single line which is at right-angles with the perpendicular from A to L is the foundation of a great part of the ordinary geometry and plane trigonometry. While the assumption that the second class consists of more than one line leads to a newer geometry, whose results are also free from contradictions.[5] This newer geometry was called non-Euclidean geometry by Gauss, imaginary geometry by Lobatschewsky, and absolute geometry by Johann Bolyai.[6]

It is certainly of interest to learn what some of the foremost mathematicians have said with respect to this geometry. Professor Sylvester said in regard to Lobatschewsky's work: —

"In quaternions the example has been given of algebra released from the yoke of the commutative principle of multiplication — an emancipation somewhat akin to Lobatschewsky's of geometry from Euclid's noted empirical axiom."

Professor Cayley said: —

"It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory wherein this axiom was not assumed to hold good, or, say, a system of non-Euclidean plane geometry."

Another very eminent mathematician, Professor Clifford, in speaking about the same work, said: —

"What Vesalius was to Galen, what Copernicus was to Ptolemy, that was Lobatschewsky to Euclid."

Something of the nature of this geometry may be inferred from a few of its theorems which differ from the corresponding theorems of the ordinary geometry. In addition to the important theorem that the sum of the internal angles of a plane finite triangle is less than two right-angles, it is proved that if we have given a line (L) and a perpendicular (B) to L, the parallels to L through points on B will make angles with B varying from $\frac{\pi}{2}$ to 0; so that we can draw through B a parallel to L making any given angle with B.[7]

The locus of a point at a constant distance from a straight line is a curved line.[8]

The areas of two plane triangles are to each other in the ratio of the excesses of two right-angles over the sums of their angles.[9]

We proceed now to some observations on the second property of Euclidean space mentioned above, viz., that space is an infinite continuity of three dimensions. We shall not take up the question of the infinitude of space nor Riemann's distinction between

[1] Cantor's Vorlesungen über Geschichte der Mathematik, Vol. I., p. 224.

[2] Encyclopædia Britannica, Vol. VIII., p. 657.

[3] Frischauf's Absolute Geometrie, pp. 14, 15.

[4] Briefwechsel zwischen Gauss und Schumacher,— especially Vol. II., pp. 268-271.

[5] Lobatschewsky's Theorie der Parallellinien, Art. 22.

[6] Frischauf's Absolute Geometrie, Art. 13.

[7] Lobatschewsky's Theorie der Parallellinien, Art. 23.

[8] Frischauf's Absolute Geometrie, p. 18.

[9] Frischauf's Absolute Geometrie, p. 50.

infinite and undounded as applied to space, we shall content ourselves with a few remarks on the number of dimensions of space.

In ordinary geometry we say that the limit or boundary of a solid is a surface, the limit of a surface is a line, the limit of a line is a point, while the point is indivisible. The same thought is expressed in other words when we say a solid has three dimensions, a surface two, a line one, while a point has no dimensions. Although the question of three dimensions of space has engaged the attention of many philosophers, no one has succeeded, to the present, to give a deep reason which is not based upon our experiences why after three passages over the limits (beginning with a solid) we should arrive at the indivisible.[1] Our inability to conceive solids or figures of more than three dimensions does not disprove their existence. If we imagine a world of two dimensions, in which all things consist of two dimensional figures, in which the inhabitants are so constituted that they can receive impressions only from things in the surface which constitutes their universe, and if we consider how unthinkable to such beings might appear figures of three dimensions, we may perhaps be prepared to admit the possibility of a space of more than three dimensions.[2]

The relations of algebra and geometry are such that an equation involving n unknowns ($n \overline{\gtrless} 3$) finds its geometric interpretation in a space whose dimensions are equal to the number of unknowns in the equation. The dual (algebraic and geometric) solution of algebraic equations enhances greatly their value and interest Algebra does not restrict itself to a fixed number of unknowns. The question whether there is a corresponding practical geometry of a space whose dimensions are not fixed is of the greatest interest. We shall designate such a space by E_n, $(0 \overline{\gtrless} n \overline{\gtrless} \infty)$, hence E_n contains all the points of this space.

In constructing a geometry for E_n it is necessary to select a set of axioms. These axioms must be so chosen that when E_n becomes an E_a $(0 \overline{\gtrless} a \overline{\gtrless} 3)$ this geometry will lead to results harmonizing with our experiences. We proceed to give a few of the assumptions from an approved work on n-dimensional space.[3]

Through each point pass many E_{n-1} having the following properties: —

Through point of an E_{n-1} pass many E_{n-2} on which the E_n may be moved; by this motion the E_{n-1} may be made to occupy completely its first position, while the individual points have changed their positions.

In each E_{n-2} there are many E_{n-3} on which the E_{n-1} may be rotated in itself. By this rotation each point will describe a closed curve.

Starting with such assumptions, a geometry is constructed by collecting and classifying theorems which rest ultimately upon them. It is perhaps worthy of remark that attempts have been made to prove the impossibility of a fourth dimension.[4]

As the main object of this article is the presentation of the non-Euclidean geometry of two dimensions, we proceed to develop the foundations on which rests a still more general two-dimensional geometry than the one noted in the fore-part of this paper. The understanding of the following processes will demand some mathematical attainments beyond what is required to appreciate the preceding. The formula which we desire to use is given in Killing's Nicht-Euklidische Raumformen, p. 14. We shall here give a simple outline of its development, referring the reader to that work for the rigorous proofs of some of our statements. We give here two almost axiomatic theorems which we shall need later.

To a triangle whose sides are all infinitesimals all the principles of the ordinary geometry and plane trigonometry apply, independent of the axiom of parallels. If one angle of a triangle becomes an infinitesimal while the others remain finite, the ratio of the sides including the infinitesimal angle has unity for its limit.

Given a triangle with a constant finite side (c) and a constant adjacent angle β, while the other adjacent angle (α) is infinitesimal. The side (a) opposite α must also be infinitesimal. The lines which divide this α into n equal parts also divide a into n equal parts. Hence the ratio $\frac{a}{\alpha}$ depends not upon α (α remaining infinitesimal) but upon c and β. The limit of this ratio when $\beta = \frac{\pi}{2}$ and α is in the act of vanishing is denoted by $f(c)$. For any other value of β this limit is $\frac{f(c)}{\sin\beta}$. Let c increase by an infinitesimal h, and $\frac{a'}{\alpha} = \frac{f(c+h)}{\sin\beta}$. Since $\frac{\sin\beta}{\alpha} \times \frac{a'-a}{h}$ has a finite limit when h is in the act of vanishing, its equal, $f'(c)$, must have the same limit. We may suppose a triangle formed by keeping β and c constant while α increases to a finite angle. We thus obtain a triangle in which a, β, α, c are finite. We will call the third side and the third angle b and γ, respectively. In this triangle we may let α undergo an infinitesimal increase, $d\alpha$, at the same time a, b, γ will increase by da, db, $d\gamma$, respectively. This increase of the triangle is a triangle like the one just considered, and the formulas obtained are directly applicable to it. The following formulas can easily be proved : —

$$\frac{da}{d\alpha} = \frac{f(b)}{\sin\gamma}, \quad \frac{d\gamma}{d\alpha} = -f'(b), \quad \frac{db}{d\alpha} = \cos\gamma. \tag{1}$$

The first has been found in the preceding triangle. From it we also obtain —

$$\frac{\sin(\pi-\gamma)}{d\alpha} \times \frac{a'-a}{h} = f'(b);$$

but

$$\frac{a'-a}{h} = \frac{\sin[\pi - (\pi - \gamma + \gamma + d\gamma)]}{\sin\gamma},$$

hence the second equation. The third follows directly after drawing perpendicular from γ upon the side $b + db$.

From equations (1) we obtain easily —

$$\frac{f'(b)\,db}{f(b)} = -\frac{\cos\gamma\,d\gamma}{\sin\gamma}.$$

Intergrating this —

$$\log f(b) = -\log\sin\gamma + \log C.$$

When $\alpha = 0$, it follows that $b = c$, $\gamma = \pi - \beta$. From this we find $\log C$, and the equation takes the form

$$f(b)\sin\gamma = f(c)\sin\beta.$$

Differentiating, α and c being regarded constant, we obtain

$$f'(b)\sin\gamma\,db + f(b)\cos\gamma\,d\gamma = f(c)\cos\beta\,d\beta.$$

Substituting from (1), remembering that β, b takes the places of α, a, there results

$$f(a)f'(b) - f(b)f'(a)\cos\gamma = f(c)\cos\beta.$$

Hence, cyclically,

$$f(a)f'(c) - f(c)f'(a)\cos\beta = f(b)\cos\gamma.$$

Multiplying the last by $f'(a)$ and subtracting from the preceding there results,

$$f(a)[f'(b) - f'(a)f'(c)] = f(c)[1 - \{f'(a)\}^2]\cos\beta.$$

Combining with this the analogous equation

$$f(c)[f'(b) - f'(a)f'(c)] = f(a)[1 - \{f'(c)\}^2]\cos\beta.$$

By division,

$$\frac{\{f(a)\}^2}{1 - \{f'(a)\}^2} = \frac{\{f(c)\}^2}{1 - \{f'(c)\}^2}.$$

Since a and c are independent, the members of this equation must be constant. Hence the important equation —

$$\frac{\{f(c)\}^2}{1 - \{f'(c)\}^2} = k^2$$

[1] Killing's Nicht-Euklidischen Raumformen, p. 64.

[2] A short romance, entitled "Flatland," depicts the difficulty an inhabitant of a two-dimension world (a square) had to conceive of three-dimensional space, even after he had acquired some idea of a one-dimension, or line, world. The book is published by Roberts Brothers, Boston, Mass.

[3] Killing's Nicht-Euklidischen Raumformen, p. 65.

[4] Max Simon, Zu den Grundlagen der nicht-euklidischen Geometrie, p. 26.

In which k is some constant. Substituting from (1) we obtain

$$\frac{1}{k^2} = \frac{da^2 - d\gamma^2}{da^2 \sin \gamma}.$$

By making $\frac{1}{k^2} = 0$, we obtain $da = d\gamma$, which, from the triangle in which these occur, is equivalent to saying, the sum of the angles of a plane triangle is constant and equal to two right-angles. This hypothesis leads to the Euclidean, or parabolic, geometry, making $\frac{1}{k^2} < 0$ makes $d\gamma > da$, which shows that the sum of the angles of a plane triangle is less than two right-angles, and leads to Lobatschewsky, or hyperbolic, geometry. Finally, the hypothesis $\frac{1}{k^2} > 0$ makes $da > d\gamma$ and indicates that the sum of the angles of a plane triangle is greater than two right-angles. This gives rise to the elliptic geometry. The last is divided into two divisions — the single elliptic geometry and the double elliptic geometry. The names parabolic, hyperbolic, single elliptic, and double elliptic were applied to these spaces by Klein. The last two kinds of space are nearly alike. Euclidean geometry may be regarded as the common limit of the hyperbolic and the elliptic geometry. Considerations similar to the preceding lead to four kinds of n-dimensional space, and hence there are four kinds of n-dimentional geometry.

ALTAKAPAS COUNTRY.

BY JOHN GIFFORD, SWATHMORE COLLEGE, PA.

IN the southern part of Louisiana there is an interesting region called the "Altakapas Country." It was once inhabited by a tribe of Indians of that name. They have the reputation of having been cannibals, but the later generations were peaceful and industrious. A few of them, they say, still exist and are famous for the skilful manner in which they make a peculiar kind of basket-work. Specimens of this may be seen in the museum of the Tulane University of Louisiana.

Roughly speaking, the region referred to embraces the land bordering the Gulf, west of the Atchafalaya and east of the Mermentan River. There is some discussion as to the extent of the country known by that name. As ordinarily used the term is elastic, but in a map printed in 1826 it includes all of what was then known as La Fayette, St. Mary's, and St. Martin's parishes and what is now known as Vermillion, La Fayette, St. Martin's, and St. Mary's.

Excepting five islands to which I shall refer later, this country is low, level, and rich. It is a part of the alluvium of the delta, which is intersected by many bayous, the arteries of Louisiana. The Atchafalaya is sometimes called "Old River," and was once no doubt the bed of the Mississippi. To-day it is reddened by the water from the Red River, in the mouth of which it begins. It is now perhaps the largest collateral artery of the main trunk. It was once clogged by an enormous raft, which was removed by the State in 1835. According to LeConte, it "was a mass of timber eight miles long, seven hundred feet wide, and eight feet thick. It had been accumulating for more than fifty years, and at the time of its removal was covered with vegetation, and even with trees sixty feet high."

The Altakapas country consists of tilled lands, low meadows, and sea-marshes. The thriftiest of the first extends along that tortuous, sluggish stream called Bayou Teche. It is very rich and well cultivated, and by many is considered the garden-spot of the State. The banks of the Teche are lined by beautiful sugar plantations with old-time palatial residences and many modern refineries. Cane is there worked, and sugar and molasses manufactured according to the latest scientific methods. Enormous quantities of sugar, molasses, rice, cotton fibre, oil, and meal, and cypress lumber are shipped from this region. Even the moss on the trees is the source of an income of no little consequence.

This bayou begins in a network of streams in the Red River country and empties into the Atchafalaya below Grand Lake. In few places in the world will you meet with such scenery. A trip down the Teche from St. Martinsville, a quaint town grey with age and "finished" long ago, once called "the little Paris, the land of Evangeline," on a sugar-packet is claimed by many, for scenery of its kind, to be unrivalled outside of Louisiana.

West of the Teche are miles of meadow-land, where many herds of horses and cattle pasture. Southward bordering the bays and Gulf is a region of sea-marshes and floating prairies.

In the midst of this marsh, near Vermilion and Atchafalaya Bays, there is a chain of five islands, the highest land in lower Louisiana. The most western is called sometimes Miller's, sometimes Orange, and sometimes Jefferson's Island. It is the centre of Joseph Jefferson's famous plantation. The second is called Petite Anse or Avery's Island, where the Avery salt mine is located, the like of which, they say, does not exist in this country. The third is Week's Island, the fourth Cote Blanche, and the last Belle Isle.

The fact that five islands exist, much different from the surrounding country, of a different formation, in a straight line, about six miles apart, in the Mississippi Delta is curious. But stranger still the core of Avery's Island is a mass of rock salt of the purest kind, the only impurity, in fact, is .120 per cent of gypsum.

While prospecting for the opening of another mine, they found the bones of the mastodon, giant sloth and perhaps of other extinct animals in layers of material of a peaty nature. Here, also, were found beautiful potsherds and kitchen middens of the Indians who once lived there. There were also indications, I was told, that the Indians knew of the presence of this salt, although, according to Dr. Hilgard, it was not discovered by the whites until 1862. The bones and potsherds which were found there are now in the museum of the Tulane University of Louisiana.

To scientists and sightseers these mines are well worthy a visit, but unfortunately are rather inaccessible. It is easiest reached from New Iberia on the Southern Pacific Railroad. There is a freight train running to the mines, which carries a passenger car. This remains, however, only long enough to collect the freight, which is seldom more than thirty minutes. There is only one train daily. The wagon-road is dangerous at times and never pleasant for vehicles owing to much mud, bad bridges and a pole-road over the marshes. The best way to reach it is on horseback, and for this purpose the Acadian ponies have no equal. They have a peculiar gait, faster than a fast walk, and lift their feet in a quick peculiar manner, which comes, they say, from pulling their feet quickly out of miry places.

The island is visible a long way off, and owing to the contrast with the surrounding country is very striking and prominent. The soil is pure sand and clay, in places mixed to form a loam.

To enter the mine you are apparently instantly dropped down a shaft one hundred and seventy feet deep. You are then in a huge cake of salt resembling ice. The weight above is supported by huge pillars of salt. Enormous quantities have been removed and the supply seems exhaustless. In places it is as clear and transparent as ice, in others granular, in others dark in color, and in others in irregular waves as though contorted by pressure. Here and there are pockets in which beautiful cubical crystals may be found, some of which the writer collected were $1\frac{1}{2} \times 1\frac{1}{2} \times 1$ inches in size.

Although it affords ventilation, they have been troubled by a slight cave, which of course gradually washes larger in size, and a fine sand is thus washed into the mines.

In the Smithsonian Contributions, Vol. XXIII., Dr. Eugene Hilgard has described this formation in a paper entitled "Geology in Lower Louisiana and the Salt Deposit of Petite Anse Island."

One of the other islands borders on the bay, where there is a bluff from which the formation may be studied.

Over in the neighboring parish of Calcasieu, near Lake Charles, there is a bed of sulphur which promises to become an important industry.