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ON THE FUNDAMENTAL HYPOTHESES OF ABSTRACT DYNAMICS.[1]

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THE formally recognized axioms of abstract dynamics employed by most writers are the three Laws of Motion enunciated by Newton in the "Principia," not always in the form given them by Newton, but in some form or other. It is obviously important that such axioms should be precise in their enunciation, independent of one another, sufficient for the deduction of all propositions applicable to natural forces generally, and as few as possible.

These axioms are sometimes regarded as constituting a definition[2] of force. As defining force, however, they are not consistent with one another; for momentum being a relative conception, i.e., having magnitude and direction which vary with the point by reference to which velocity is specified, force, if defined by the first and second laws, must also be a relative conception. And it follows that the third law cannot in general hold; for it is easy to show that if it hold for one point of reference, it cannot hold for another having an acceleration relative to the first.

The axioms are thus statements about the action of force, force being assumed to be already a familiar conception. As applicable to the translation of bodies, they may be regarded either as hypotheses verified by the deductions made from them, or as generalizations established by direct though rough experiments. When, however, we come to study the effect of force in changing the rotation of bodies or their state of strain, we assume the laws of motion to hold for the small parts (particles or elements) of which we imagine the bodies to consist. And therefore, as forming the basis of dynamics as a whole, they must be regarded as hypotheses. In either case it is necessary to note that both the popular and the scientific conceptions of force ascribe to it a magnitude and direction quite independent of the point of reference which may be used in specifying the motion of the body on which it acts.

I. The Precision of the Laws of Motion.

Owing to this non-relative character of force, it is obvious that the first and second laws of motion can hold only provided the motion of bodies be specified relatively to certain points. In omitting the mention of these points, Newton's laws are somewhat lacking in precision; and it is important to determine what the points are.

As, according to the first law, two particles which are both free from the action of force must have uniform velocities, relatively to the unspecified point of reference, each must have a uniform velocity relatively to the other. Hence the first law, as pointed out by Tait,[3] holds relatively to any particle on which no forces act.

As, according to the second law, the acceleration of either a particle of finite mass acted upon by no force, or a particle of infinite mass acted upon by no infinite force, must be zero relatively to the unspecified point of reference, this law must hold relatively to all such particles.

But such particles are fictitious. To bring the second law within the region of practical application, we must find accessible points by reference to which it holds. This may readily be done; for it is easy to prove it to hold for a particle acted upon by given forces, relatively to any other particle, with respect to which, but for the action of these forces, the former would have no acceleration. Thus, as is usually assumed, the acceleration, relative to a point of the earth's surface, of a body situated at that point and at rest or in uniform motion relatively to it, except in so far as its motion may be modified by given forces, may be determined by the application of the second law.

It is interesting to note that this was the point of reference employed by Newton in the experiments made by him to verify the third law. In these well-known experiments[4] on the impact of spheres, the spheres were suspended by strings, and impact was made to occur when the spheres occupied their lowest positions. Their velocities before and after impact were taken to be proportional to the chords of the arcs (corrected for resistance of air), through which they had fallen, or were found to rise respectively. Hence the acceleration of a freely falling body was assumed to be vertical; and the point of reference was consequently the point of the earth's surface at which the experiments were made. Also at the instant of impact, the spheres were passing through their positions of zero acceleration relatively to this point. Hence the equal and opposite changes of momentum observed were specified by reference to a point with respect to which, apart from the action of the stress due to impact, the impinging spheres had no acceleration.

As the third law asserts merely the equality and opposition of two forces, it must hold for all points of reference; or rather it is independent of points of reference.

It follows that besides the points mentioned above, with respect to which the second law holds, there is, in the case of a system of particles, free from the action of external force, another, viz., the centre of mass of the system. For this point may be shown by the aid of the third law to have no acceleration relatively to any point, by reference to which the second law holds.

It may easily be proved that the stress between two particles is proportional to the product, by the sum of their masses into their relative acceleration; and that consequently, if one of the particles be of infinite mass, the stress is proportional to the mass of the other multiplied by the relative acceleration. Hence if, in applying the second law of motion, a particle of infinite mass be chosen as point of reference, all the forces acting on a system of particles, both external and internal, may be regarded as exerted upon them by the particle of infinite mass.

[1] Abstract of the presidential address to the Mathematical and Physical Section of the Royal Society of Canada, at the meeting held May, 1892.

[2] Maxwell's Matter and Motion, Art. XL.

[3] Properties of Matter (1885), p. 92.

[4] Principia: Scholium to Axiomata.

2. Independence of the Laws of Motion.

Maxwell[1] maintains that "the denial of Newton's first law is in contradiction to the only system of consistent doctrine about space and time which the human mind has been able to form." If this be so, it must be possible to deduce the law from the doctrine of space and time, and it cannot be held to be hypothetical in character. Maxwell's argument is as follows: "If the velocity [of a body freed from the action of force] does not remain constant, let us suppose it to vary. The change of velocity must have a definite direction and magnitude. By the maxim that the same causes will always produce the same effects, this variation must be the same, whatever be the time or place of the experiment. The direction of the change of motion must therefore be determined either by the direction of the motion itself or by some direction fixed in the body. Let us, in the first place, suppose the law to be that the velocity diminishes at a certain rate. . . . The velocity referred to in this hypothetical law can only be the velocity referred to a point absolutely at rest. For if it is a relative velocity, its direction as well as its magnitude depends on the velocity of the point of reference. . . . Hence the hypothetical law is without meaning unless we admit the possibility of defining absolute rest and absolute velocity."

This argument, which is endorsed by Tait,[2] may be used to prove Newton's law also to be without meaning. For this purpose all that is necessary is to substitute *displacement* for *velocity* or *motion*, wherever these words occur in the above quotation, and *changes* for *diminishes*. The argument is thus transformed into one equally good or bad, in favor of the cessation of motion on the cessation of the action of force, as against Newton's law.

The fallacy — for the argument would thus appear to be fallacious — seems to lie in the incomplete recognition of the relativity of the law of motion under consideration. Thus, when, in the second sentence of the above quotation, Maxwell says: "The change of velocity must have a definite magnitude and direction," he forgets that its magnitude and direction must vary with the point of reference. And the whole argument turns upon this asserted definiteness.

While the first law must be considered incapable of deduction, its right to formal enunciation among the fundamental hypotheses of dynamics has often been disputed on the ground of its being a particular case of the second law. This must be admitted; and its separate enunciation must therefore be pronounced illogical.

There is one objection, however, which may perhaps be urged against the omission of the first law, viz., that Maxwell[3] and other authorities hold that this law, "by stating under what circumstances the velocity of a moving body remains constant, supplies us with a method of defining equal intervals" of time. As no such statement is ever made about the second law, it would thus appear that the omission of the first would leave us without a basis for the measurement of time.

This objection, however, is easily met. For, first, the second law supplies us with more methods of defining equal intervals of time than the first law. In addition to the definition given by the latter, it tells us, for example, that those intervals are equal in which a body acted upon by a constant force undergoes equal changes of velocity.

Second, both laws assume that equal intervals of time have already been defined. So far as power of defining is concerned, therefore, they give us nothing that we did not possess before their enunciation. The only advance in time-measurement which we owe them is that they show us how to construct time-pieces which will mark off for us the intervals assumed to be equal in their enunciation.

Third, the intervals assumed equal in the enunciation of these laws are not known to be equal. What they assume is therefore nothing more than a conventional time-scale; and what they give us is nothing more than certain methods of securing accurate copies of this scale.

And, fourth, both of these laws may be enunciated so as to retain all their dynamical significance, and yet make no reference to the measurement of time, by adopting as the definition of velocity not distance traversed per unit of time, but the distance traversed while the earth (or, better, a certain ideal earth) rotates through a certain angle relatively to the fixed stars. Enunciated in this way these laws assume no definition of equal intervals of time, and can consequently supply us with no such definitions.

Newton's second law asserts that the acceleration produced in a body by a force is directly proportional to the force and has the same direction; and as the assertion is without restriction, the law implies that the effect of the force is the same, whatever the motion of the body may be and whatever other forces may be acting upon it. Many writers regard the latter implied part of the law as being the only hypothetical part. They therefore make it the second law of motion and attempt to deduce the former part from it, the argument being that since any number n of equal and codirectional forces will produce in a body an acceleration n times as great as that produced by one, the acceleration produced in a body must be proportional to the force producing it. It is here assumed, however, that n equal forces in the same direction are equivalent to a single force of n times the magnitude. Thus the explicitly asserted portion of Newton's second law cannot be deduced from the implied portion except by the aid of an additional hypothesis; and the law as a whole must therefore be regarded as hypothetical.

The third law is supposed to have been deduced from the first by Newton himself. Maxwell[4] appears to hold this view; Lodge[5] declares his adhesion to it; and Tait[6] says the third law "is very closely connected with the first." Newton's discussion[7] of the third law, in which he is supposed to make this deduction, consists of two parts. He first shows by the experiments referred to above, that the law applies to the case of the stresses between bodies pressing against one another; and he then extends it by the aid of the first law to gravitational stresses, and by the aid of further experiment to magnetic stresses as well. In this extension he does not say that he is building upon the results of his experiments on impact, but it seems obvious that he does so. Maxwell summarizes his argument admirably in the following words: "If the attraction of any part of the earth, say, a mountain, upon the remainder of the earth, were greater or less than that of the remainder of the earth upon the mountain, there would be a residual force acting upon the system of the earth and the mountain as a whole, which

[1] Matter and Motion, Art. XLI.

[2] Ency. Brit., 9th Ed., Art. Mechanics, § 298.

[3] Matter and Motion, Art. XLIII.

[4] Matter and Motion, Art. LVIII.

[5] Elementary Mechanics (1885), p. 56.

[6] Properties of Matter (1885), p. 103.

[7] Principia: Scholium to Axiomata.

would cause it to move off with an ever-increasing velocity through infinite space. This is contrary to the first law of motion, which asserts that a body does not change its state of motion unless acted upon by external force." That this argument is based upon the assumption of the equality of the action and reaction between bodies pressing against one another, seems to follow from the consideration that otherwise the "residual force," due to the possible inequality of the action and reaction of the gravitational stress between the mountain and the remainder of the earth, might be regarded as neutralized by an opposite inequality in the action and reaction of the stress at their surface of contact. Even, therefore, if Newton's extension of his experimental result to forces acting at a distance were regarded as valid, the third law could not be regarded as deduced from the first. It would only be shown to be but partially hypothetical. But since, in the present state of dynamics, the laws of motion must be regarded as applicable to particles, Newton's argument, though valid when they were considered applicable to extended bodies, can no longer be admitted; for the uniformity of the motion of a body free from the action of external force is itself a deduction, which can be made only by assuming the third law in its most general form.

3. Sufficiency of the Laws of Motion.

The best test of the sufficiency of the laws of motion is the question, Can they give by deduction the greatest of all physical laws, the conservation of energy? This law may be proved, by the aid of the second and third laws of motion, to hold in the case of any system of particles which is neither giving energy to, nor receiving energy from, external bodies, provided the stresses between the particles act in the lines joining them and are functions of their distances. It has also been proved by experiment to hold in a very large number of cases in which the laws of the forces acting are unknown, the energy disappearing in one form and the energy appearing simultaneously in another form being measured. The amount of such experimental evidence is so large that no doubt is now entertained that the law of the conservation of energy is applicable to all natural forces. Hence the fundamental hypotheses of dynamics should either include this law or give it by deduction.

Although many writers state that this law may be deduced from the laws of motion, Lodge[1] is the only one, so far as I am aware, who claims to make the deduction. This he does in a passage beginning as follows: "All this, indeed, in a much more complete and accurate form — more complete because it involves the *non destruction* of energy, as well as its non-creation — follows from Newton's third law of motion, provided we make the assumptions (justified by experiment)," etc. It is unnecessary to quote farther; for when assumptions justified by experiment are called in to the aid of the third law, additional fundamental hypotheses are thereby selected.

The second law of motion enables us to take the first step in the deduction of the conservation of energy. The proof is so well known that I may simply cite that given by Thomson and Tait,[2] resulting in the familiar equation: —

$$\Sigma\,(X\dot{x}+Y\dot{y}+Z\dot{z})=\Sigma\,m\,(\ddot{x}\dot{x}+\ddot{y}\dot{y}+\ddot{z}\dot{z}),$$

in which the first member represents the rate at which work is being done by the forces acting on the particles of a system, and the second is equal to the rate at which the kinetic energy of the system is being increased. It is usually called the equation of *vis viva*, and, having been deduced from the second law of motion alone, is applicable to all forces, whether conservative or not.

Newton gave this result in the Scholium to the Laws of Motion in a statement which may be paraphrased thus: Work done on any system of bodies has its equivalent in work done against friction, molecular forces, or gravity, together with that done in overcoming the resistance to acceleration. Thompson and Tait point out expressly[3] that this statement of Newton's, which, owing to the form he gave it, is often referred to as his second interpretation of the third law of motion, is equivalent to the equation given above. Nevertheless, it has been interpreted as being little less than an enunciation of the law of the conservation of energy itself.[4] Thus Tait[5] says it "has been shown to require comparatively little addition to make it a complete enunciation of the conservation of energy;" and "What Newton really wanted was to know what becomes of work which is spent in friction." Besant[6] takes the same view.[7] These writers seem to claim that Newton's statement is equivalent to what Thomson and Tait call "the law of energy in abstract dynamics," viz., "The whole work done in any time on any limited material system by applied forces is equal to the whole effect in the forms of potential and kinetic energy produced in the system, together with the work lost in friction." Of this it may certainly be said that what it wants to make it a complete enunciation of the conservation of energy is a statement as to what becomes of the work spent in friction.

Compare this, however, with Newton's statement, as paraphrased above, and it is at once obvious that what the latter wants to make it a complete enunciation of the conservation of energy, is a statement as to what becomes not only of work spent in friction, but also of work done against molecular forces and gravity, and of work done in overcoming the resistance to acceleration. Newton may possibly have known all this, but he does not say so; and we must therefore hold his statement to be, as Thomson and Tait point out, merely a verbal expression of the equation given above. The question of the interpretation of Newton's statement is of more than mere historical interest; for if it would bear the interpretations which have been put upon it, the law of the conservation of energy would be capable of being deduced from the second law of motion alone.

To pass from the equation of *vis viva* to the law of the conservation of energy, we require to know that the work done during any change of configuration of a system of particles acted upon by natural forces depends only upon the changes in the positions of the particles, and not upon the paths by which or the velocities with which they have moved from the old positions to the new. Helmholtz[8] showed that this deduction may be "based on either of two maxims, either on the maxim that it is not possible by any

[1] Elementary Mechanics (1885), p. 82.

[2] Treatise on Nat. Phil. (1879), Vol. I., Part 1, p. 269.

[3] Treatise on Nat. Phil. (1879), Vol. I., Part 1, p. 270.

[4] This address was written before I had seen Professor W. W. Johnson's paper on "The Mechanical Axioms, or Laws of Motion" (Bull. N. Y. Math. Soc., Vol. I., No. 6, March, 1892).

[5] Properties of Matter (1885), p. 104, and Recent Advances in Physical Science (1876), p. 38.

[6] Dynamics (1885), p. 49.

[7] Garrett (Elementary Dynamics, 1886, p. 47) goes so far as to say that Newton's statement "is nothing more nor less than the enunciation of the great principle of the conservation of energy."

[8] On the Conservation of Force (1847): Taylor's Scientific Memoirs. Nat. Phil. (1853), p. 114.

combination whatever of natural bodies to derive an unlimited amount of mechanical force [energy], or on the assumption that all actions in nature can be ultimately referred to attractive or repulsive forces, the intensity of which depends solely upon the distances between the points by which the forces are exerted." He showed also that it was immaterial which of these maxims was assumed, as the other could be at once obtained from it. How by the aid of either of these hypotheses we pass from the equation given above to the law of the conservation of energy is of course well known. The point to which it seems necessary to draw attention is that some hypothesis is required, and that either of these is sufficient for the purpose.

As the second of Helmholtz's maxims is simply an extension of the third law of motion, and as Newton's three laws have obtained such wide usage, it would seem to be desirable to adopt the second maxim as a fourth law of motion. Were we to select the first maxim, it would be necessary to re-cast our fundamental hypotheses altogether.[1] Possibly it might be advantageous to take this course, to make, as Tait[2] suggests, the laws of the conservation and the transformation of energy our fundamental hypotheses, and to banish the conception of force to the limbo of once useful things. But if Newton's laws are to be retained, they should be supplemented by the second of Helmholtz's assumptions.

It is at once obvious that this fourth law will, like the third, be independent of points of reference; and it follows that the law of the conservation of energy will hold relatively to all points by reference to which the second law holds. This conclusion is inconsistent with Newcomb's assertion[3] that this law "assumes that we refer the motions of all the bodies whose energy is considered to some foreign body of infinite mass, from which emanate the forces which give motion to the system." According to the above, this law may of course be expressed relatively to a particle of infinite mass, and, if thus expressed, the forces which give motion to the system may be supposed to emanate from that particle. But it may also be expressed relatively either to a particle of finite mass free from the action of force, or to the centre of mass of the system itself whose energy is conserved.

4. Reduction of the Laws of Motion.

Finally, the four laws of motion may obviously be reduced to two. The first has already been seen to be a particular case of the second. The third is involved in the fourth; for when it is asserted that natural forces are attractions or repulsions, it is implied that their action and reaction are in opposite directions, and when it is asserted that they may be expressed as functions of the distances of the particles between which they act, it is implied that their action and reaction are equal. The four laws thus reduce to two, which may be enunciated somewhat as follows:—

The Law of Force. — Relatively to any particle free from the action of force, the acceleration produced in another particle by a force is proportional to the force and has the same direction.

The Law of Stress. — Natural forces may be considered to be attractions or repulsions whose magnitudes vary solely with the distances of the particles between which they act.

[1] Many writers illogically select the first maxim as a fourth law. See Professor Johnson's paper cited above; also my Kinematics and Dynamics, § 436.

[2] Ency. Brit., 9th Ed., Art. Mechanics, § 291.

[3] Phil. Mag., Ser. 5, Vol xxvii. (1889), p. 116.

THE GREAT LAKE BASINS.

BY P. J. FARNSWORTH.

THE problem of the origin of the Great Lakes has for a long time engaged the attention of the scientists, who have come to a variety of conclusions, none of them very satisfactory. Subsidence, ice action, glacial scooping, and President Chamberlin's theory that they were hollows made by accumulating ice bending down the earth's crust.

An article in *Science* of June 3 presents a more plausible theory, that they are vallies of erosion, made by some great river, giving as evidence the map of Dr. Spencer, pointing out the discoveries and probable deep pre-glacial channels leading into the St. Lawrence and the Atlantic. Professor Spencer, in his paper on High Continental Elevations, read at the Scientific Association at Toronto, 1889, sums up by saying, "The lake basins are merely closed-up portions of the ancient St. Lawrence valley and its tributaries." "The lake basins are all excavated out of Palæozoic rocks except a part of that of Lake Superior."

If we go back in geologic history to Azoic times we find that the first emergence of the continent was the V-shaped land around Hudson's Bay, an open sea below it. Next, an emergence of a point below the V and a line of height extending along the lower side of what we call the river and gulf of St. Lawrence. A sea or strait extended round the primitive land from the Atlantic to the Arctic Ocean on the north-west. After the elevation of the trough at the north-west, an inland sea was left covering Superior, Michigan, Huron, and Ontario, leading into the St. Lawrence Gulf. In time there was elevation and subsidence and flexion of strata, as pointed out by Professor Spencer, and the great basins were left as interior seas. There was a large watershed to the north that compelled an overflow, that made its way in the deep channels that have been discovered, at some time out of Ontario, across New York, then, if there was continental elevation, making the deep channels down the valley of the St. Lawrence and far out into the Gulf. Lake Champlain was a pool in a fissure of the Azoac world, that was connected with the open channel in the Archean land.

The ice period so obstructed the old outlet that when it was melting, the superfluous waters of the great basins were poured into the Gulf of Mexico through the Illinois and Wabash rivers. When the ice disappeared, the old outlet had become obstructed by flexions of strata and mountains of drift. It is evident that Lake Michigan had a channel through Georgian Bay, and thence into Ontario. It is not yet apparent where the deep channel for the waters of Superior came in, or that it had any such. It has an insignificant but sufficient outlet through the St. Mary's River. Michigan and Huron reach Ontario over the St. Clair flats and through the shallow trough that holds Lake Erie, which probably is of post-glacial age, and then into Ontario down the hill that is being cut back by the falls of Niagara.

The great lakes were deep seas before the world was cold enough for ice, and were great basins before glaciers were possible.

One could hardly conceive how glacial ploughing coming from the north or north-east could make chasms at such angles to each other. In regard to cut of channels of erosion, it would require a river from the south-west and north-west, from Michigan and Superior, of such magnitude that great valleys or traces of them would be left. Lake Superior is 360 miles long and 150 miles wide in some places, with a