SCIENCE

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THE EVOLUTION OF ALGEBRA.[1]

In considering the possible subjects for an address on this occasion, it has seemed to me that a half-hour might be agreeably spent in a brief survey of the progress, or evolution, of algebra from its earliest known beginnings to the present time.

The realm of mathematics may be classified, in a general way, into (1) Arithmetic, or the theory of numbers, (2) Algebra, (3) Geometry, though sharp dividing lines cannot always be drawn between these departments: the last two, for instance, mutually interacting, geometry illustrating algebra, while algebra is the efficient servant of geometry, enabling it to conquer territory which it could scarcely have entered upon unaided.

The history of the development of these different branches of mathematics shows considerable diversities among them. Thus geometry reached in a short time, among the ancient Greeks, a high stage of advancement, and then became practically stationary until quite recent times, while the progress of algebra has been more in the nature of a gradual and continuous evolution. Nesselmann has recognized three stages in this development, which he designates as the rhetorical, the syncopated, and the symbolical, to which I may perhaps venture to add the "multiple," in which a plurality of fundamental units is recognized and treated. We may regard the first three as somewhat analogous to the stone, bronze, and iron ages in human history, overlapping each other, as do these, at different times and places; while the last may be compared to that age of aluminum which is perhaps dawning upon the world.

Rhetorical algebra was a process for determining the unknown quantity in an equation by a course of logical reasoning expressed entirely in words, without the use of any symbols whatever, similar to our present mental arithmetic. In course of time abbreviations of those words which constantly recurred were introduced, by the use of which the statement of the reasoning could be much shortened, it being even possible with the notation of Diophantos to approximate to the conciseness of the modern, or symbolic, method. This, however, was not done by Diophantos himself, who used his abbreviations strictly as such, and reasoned out his results in words combined with these. This method is what is designated by Nesselmann as the syncopated, and forms evidently a stepping-stone toward the symbolic, in which perfectly arbitrary symbols are employed to represent the various quantities dealt with, and no words are written out except a conjunction now and then.

The earliest traces of algebraic knowledge which have been discovered are found in Egypt, that wonderful land whose records carry us back to such a remote antiquity. Ahmes, in a papyrus manuscript, dating from about 1400 B.C., deals with certain geometric and algebraic problems, and seems to have had as good a conception of the symbolism of algebra as his successors of a much later period. Thus he had signs for $+$, $-$, and $=$, and used the character representing a heap for the unknown quantity. He seems, therefore, to have long anticipated Diophantos in the use of syncopated notation. Our knowledge of Egyptian mathematics subsequent to this time is very slight, and is gleaned from the statements of various Greek and Latin authors.

We will pass, then, at once to the Greek contributions to the development of our subject. So far as can now be ascertained, probably but little strictly algebraic work was done before the third or fourth century of our era, though opinions differ on this point. The wonderful accomplishments of Archimedes were mainly geometrical and mechanical, though he makes one remark which is equivalent to a statement regarding the roots of an equation of the third degree, which is remarkable as being, with one exception, the only known case of any consideration of such an equation until after the lapse of more than a thousand years from his time.

Thymaridas in the second century of our era is the earliest mathematician known to have enunciated an algebraic theorem. This was, however, done entirely in words, no symbol for any quantity or operation being used.

Practically the foundation of algebra was laid by Diophantos of Alexandria. But little is known of this remarkable man. Though we have his writings in Greek, he was probably not himself a Greek. The period at which he lived is in dispute, though probabilities favor the fourth century of our era. Even the spelling of his name is uncertain, there being a question as to whether the last syllable should be *os* or *es*. But whatever may be known or unknown about the man himself, his writings show a very wonderful power of analytic reasoning, especially when we consider the awkwardness of the tools with which he was obliged to work.

What strikes us at once, from our present point of view, as most hampering is the fact that he had only one symbol for the unknown, so that, in dealing with a problem which would now be solved by the aid of several such symbols, as x, y, z, etc., he was obliged to adopt some expedient, such as to make mentally such combinations and arrangements as to get along with only one. It is easy to see how much ingenuity must often have been required to accomplish this. It is a curious and surprising fact that algebraic analysis was subjected to this same limitation down to a comparatively recent period. In place of the exponents at present used to indicate the powers to which quantities are raised, Diophantos designated the square and cube of the unknown by the initial letters of the corresponding words in Greek. Thus the unknown is represented by the character ς, standing for the word ἀριθμός (i.e., number), which is also frequently written out in full; the square of the same by δ^{υ}, a contraction for δύναμις (power); and the cube by κ^{υ}, a contraction for κύβος (cube). Higher powers up to the sixth were indicated by combination or repetition of these symbols. The origin of the character for *arithmos* is uncertain; it may be the final sigma of this word, or it may be a contraction of αρ, the first two letters of the same, or it may be derived from an old Egyptian symbol for the unknown. When oblique cases of these quantities are required, the words for square and cube are written out in full, while the practice varies with regard to *arithmos*, the word being sometimes

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written out, while at other times the case-termination is written above to the right, thus $\varsigma^{\widehat{o\upsilon}}$, the symbol being also generally doubled when the signification is plural.

Diophantos indicated addition merely by juxtaposition, having no sign for plus; for minus, however, he used the sign ⋔. As a consequence, in order to avoid confusion, he was obliged to do two things; first, to designate the absolute term as so many *μονάδες*, or units, abbreviated into $\mu^{\hat{o}}$, and second, to write all the negative terms together after the positive. Thus the quantity $x^3 - 5x^2 + 8x - 1$ would be written in Diophantos's notation.

$$\kappa^{\hat{\upsilon}}\ \tilde{\alpha}\ \varsigma\varsigma^{\widehat{oi}}\ \tilde{\eta}\ ⋔\ \delta^{\hat{\upsilon}}\ \tilde{\varepsilon}\ \mu^{\hat{o}}\ \tilde{\alpha}.$$

This may be rendered more expressive if we change it by substituting Arabic numerals, and putting U for units, N for number or unknown, S for square, and C for cube: thus it becomes $C\,1\,N\,8 - S\,5\,U\,1$.

It is to be noted that Diophantos and his successors up to comparatively recent times had no conception whatever of an intrinsically negative quantity as possible. Whatever sign may have been used for minus was considered as simply indicating that one number was to be subtracted from another, and if the subtrahend were larger than the minuend no meaning was attached to the expression.

It is possible that Diophantos might have been able to escape from the limitations of his system if the letters of the Greek alphabet had not been already appropriated for the representation of particular numbers, thus precluding their use as symbols of quantity in general.

It may be of interest to give at this point specimens of the purely rhetorical and of the syncopated methods of solution. They are given by Nesselmann, and are verbatim translations from the original tongues. The first is a solution of a quadratic equation by Mohammed ibn Musa, and the second the solution of a problem by Diophantos.

A square and ten of its roots are equal to nine-and-thirty units, that is, if you add ten roots to one square, the sum is equal to nine-and-thirty. The solution is as follows: halve the number of roots, that is, in this case, five; then multiply this by itself, and the result is five-and-twenty. Add this to the nine-and-thirty, which gives four-and-sixty; take the square root, or eight, and subtract from it half the number of roots, namely, five, and there remains three: this is the root of the square which was required and the square itself is nine.

($S =$ square, $N =$ number, $U =$ unit, as above.)

To divide the proposed square into two squares: Let it be proposed, then, to divide 16 into two squares; and let the first be supposed to be one square. Thus 16 minus one square must be equal to a square. I form the square from any number of N's minus as many U's as there are in the side of 16 U's. Suppose this to be 2 N's minus 4 U's. Thus the square itself will be 4 squares 16 U's minus 16 N's. These are equal to 16 units minus 1 square. Add to each the negative term, and take equals from equals. Thus 5 squares are equal to 16 numbers. One (square) will be 256 twenty-fifths, and the other 144 twenty-fifths, and the sum of the two makes up 400 twenty-fifths, or 16 units, and each is a square.

Compare these long-drawn-out statements with their equivalents in modern notation:

First.

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 64 \\ \therefore \quad x + 5 &= 8 \\ \therefore \quad x &= 3 \end{aligned}$$

Second.

$$\begin{aligned} 16 - x^2 &= \square = (2x - 4)^2 \\ &= 4x^2 + 16 - 16x \\ \therefore \quad 16x &= 5x^2 \\ \therefore \quad x &= \tfrac{16}{5} \end{aligned}$$

The example from Diophantos evidently does not take full advantage of his notation, for the symbol for minus is not used, and in several cases the words are written out in full where abbreviations might have been employed. Further, no use is made of the symbol for equality, viz., *ι*, an abbreviation for *ἴσοι*, which is elsewhere used by the author. If the fullest use of the syncopated notation had been made, the solution would have been somewhat comparable for conciseness and brevity with the modern method, only about twice as many characters and marks being required. Solutions in this abbreviated form appear on the margins of Diophantos's manuscripts, but they are believed to have been added by some one else, and not to be due to the author himself.

The works of Diophantos, called by him "Arithmetics," deal largely with indeterminate equations and the theory of numbers. Quadratic equations are constantly solved, but only real positive results are recognized or considered; and even when there are two positive roots, only one is taken account of. One very simple case of an equation of the third degree is found.

We will turn next to the consideration of the ancient algebra of India. There lived at Patna, in India, some time in the sixth century of our era, a mathematician named Arya-Bhatta, who wrote a work treating of arithmetic, algebra, geometry, trigonometry, and astronomy. It consists in the enunciation of rules and propositions in verse. The author gives, of course in a purely rhetorical manner, the sums of the first, second, and third powers of the first n natural numbers, the general solution of a quadratic equation, and the solution in integers of some indeterminate equations of the first degree.

The only other ancient Indian mathematician of prominence is Brahmagupta, who lived in the seventh century of our era. His work is also written in verse, and is called "Brahma-Sphuta-Siddhauta," or the "System of Brahma in Astronomy." Two chapters of this work deal with arithmetic, algebra, and geometry. The treatment of algebra is purely rhetorical, and includes a discussion of arithmetical progressions, quadratic equations (only the positive roots being considered), and indeterminate equations of the first degree, together with one of the second degree.

These Indian writings are of special interest as being the sources from which the Arabs derived their first knowledge of algebra. They obtained from the Greeks before A.D. 900, thorough translations of Euclid, Apollonius, Archimedes, and others, a knowledge of geometry, mechanics, and astronomy, but had no translation of Diophantos till a hundred and fifty years later, when they had themselves already made considerable progress in algebraic analysis. From the Arabians in turn western Europe obtained, not only the decimal notation of arithmetic, but also its first knowledge of other branches of mathematics.

The first great mathematician among the Arabs is generally known by the name of Alkarismi, though this is an incorrect transliteration of only one of his names. From the title of his work, "Al-gebr we'l Mukabala," we have the name of that branch of mathematics under consideration, al-gebr signifying that the same quantity may be added to or subtracted from both sides of an equation.

Alkarismi treats the quadratic, giving geometric proofs of rules for the solution of different cases, and recognizing the existence of two roots, though he only considers such as are real and positive. He treats only numerical equations, and no distinction is made between arithmetic and algebra. This is true likewise of his Arabian successors, who, though they advanced so far as to obtain the general solution of a cubic equation, and to state such a proposition in integers of the equation $x^3 + y^3 = z^3$ is feasible, yet always adhered to the rhetorical method, and made scarcely any progress in general algebraic science. Indeed such progress was hardly possible until the introduction of symbolic methods.

The first decided steps in the direction of symbolism since the work of Diophantos were taken by a mathematician of India named Bhaskara in the twelfth century. He used abbreviations and initials to denote the unknown, a dot for minus, and juxtaposition to indicate addition. A product is denoted by the first syllable of the word for multiplication subjoined to the factors, division by the divisor being written beneath the dividend without a line between as our custom is now. The two sides of an equation are written one under the other, and explanatory records are introduced whenever it is necessary to prevent misunderstanding. Occasionally symbols are used for given as well as unknown quantities. Square, cube, and square root are denoted by the initial letters of the corresponding words. Using the Arabic, or decimal, notation, he has a character for zero, which enables him to write all his equations with all the powers of the unknown arranged in regular order on each side of the equation, certain of them being multiplied by the factor zero. This method of writing equations maintained itself till long afterwards. We have in this author a distinct advance over Diophantos and the Arabians in the introduction of various symbols for the unknown, so that several might be used in the same problem, as well as in the use of zero.

We have now to consider a new phase of algebraic progress arising from the introduction into western Europe of the works of the Arabian mathematicians. This took place through the Moors of Spain. The Greek and Arabic works were studied at the Moorish universities of Granada, Cordova, and Seville, but all knowledge of them was jealously kept from the outside world until the twelfth century, during which copies came into the possession of Christians. Up to this time Christian Europe had been almost a mathematical blank. The simple arithmetical operations they were able to perform were accomplished by the aid of the abacus, and they possessed some knowledge of astronomy and geometry, but made no progress until they were able to avail themselves of the previous labors of Greek, Hindu, and Arab, under the stimulus of which a career of advancement began which has continued to the present time. This career, however, did not begin immediately; it took several centuries to assimilate the material received from these sources, and thus to lay the foundations on which subsequent progress should rest.

During this period the rhetorical method was used in all algebraic processes, and it was not until the sixteenth century that syncopated methods were introduced, preparing the way for the symbolic methods that soon followed. Latin being the language in use, the word *res*, or *radix*, was employed for the unknown quantity, the square being called *census*, and the cube *cubus*. These words were at first written out in full and afterwards represented by *R* or *Rj*, *Z* or *C*, and *C* or *K* respectively.

The signs $+$ and $-$ are first found in a mercantile arithmetic by Johann Widmann, published in 1489, though they did not come into general use by mathematicians till a hundred years or more afterward. The most probable supposition as to their origin is that they were at first warehouse marks indicating an excess or deficiency in the contents of a package which was supposed to contain a certain definite amount. Widmann uses them purely as abbreviations, not as symbols of operation.

The first mathematical work ever printed was by Pacioli, upon arithmetic, algebra, and geometry, and marks the beginning of the syncopated stage of development in western Europe. This book appeared in 1494, just before the beginning of the sixteenth century, during which this method was in vogue. Pacioli uses initials as abbreviations for the unknown, its square and cube, and for the words "plus" and "equal," also occasionally *de* for *demptus*, instead of minus.

The sign now used for equality was introduced by Recorde in an arithmetic published in 1540. He uses also the present signs $+$ and $-$. At about the same time our present symbol for square root was introduced by Stifel, and Nicholas Tartaglia discovered the solution of the cubic equation $x^3 + px = q$, which is generally attributed to Cardan, and goes by his name. Cardan obtained the solution from Tartaglia under promise of strict secrecy, and then published it in his work "Ars Magna." Considerable advance is made in this work over anything done by his predecessors. Negative and even imaginary roots of equations are discussed, and the latter are shown to always occur in pairs, though no interpretation of them is attempted. Cardan shows that when the roots of the cubic are all real, Tartaglia's solution appears in an imaginary form. This is the first notice we find of imaginaries, and, with the exception of a similar treatment by Bombelli a few years later, and a suggestion as to their interpretation by Wallis in 1685, they were discussed by no subsequent mathematician until Euler investigated them nearly two hundred years afterward. Cardan also discovered the relations between the roots and coefficients of an algebraic equation, and the underlying principle of Descartes' rule of signs. It is to be noted that his solutions both of quadratics and cubics are geometrical.

In 1572 Bombelli published an algebra in which the same subjects discussed by Cardan are treated in about the same way, but in which a marked advance is made in notation, viz., the employment for the unknown of the symbol $\underset{\smile}{1}$, while its powers are denoted by $\underset{\smile}{2}$, $\underset{\smile}{3}$, etc. Thus he would write $x^2 + 5x - 4$ as $1\underset{\smile}{2}$ *p.* 5 $\underset{\smile}{1}$*m.* 4, *p.* and *m.* standing for plus and minus. Other writers of the same period would have written the expression thus,

$$1Z\ p.\ 5R\ m.\ 4, \text{ or } 1Q + 5N - 4.$$

Up to this time in the development of algebraic notation, whatever may have been the forms or symbols used, they were regarded simply as abbreviations for the words necessary to express the idea to be conveyed. But now the conception of pure symbolism begins to appear. Vieta, who lived in the last half of the sixteenth century, denoted known quantities by consonants and unknown by vowels, while powers were indicated by initials or abbreviations of the words *quadratus* and *cubus*. He was thus enabled to deal with several unknowns in the same problem, together with their powers. The following is a specimen of his notation. The equation $3BA^2 - DA + A^3 = Z$ he writes as

B 3 *in A quad. — D plano in A + A cubo equatur*
Z solido.

(It may be noted that he makes his equations homogeneous, and lays stress on the desirability of so doing.) This and the other examples that have been given above illustrate the great variety of notations in use during this period, no conventional system having yet been adopted to be adhered to in the main by all mathematical writers. This is, of course, an inevitable accompaniment of the formative stage of any branch of science, when a few men are working here and there in comparative isolation. This variety continued to a considerable extent throughout the seventeenth century.

In this century we arrive at a new era in mathematical development. This was brought about by the application of algebra to geometry by Descartes in the early part, and the discovery of the differential calculus by Newton and Leibnitz independently in the latter part of the century. Algebra had been used in connection with geometry before Descartes, but to him was due the discovery of the fact, that, if the position of a point be given by co-ordinates, then any equation involving those co-ordinates will represent some locus all of whose properties are contained implicitly in the equation, and may be deduced therefrom by ordinary algebraic operations.

Descartes initiated the custom, which has become fixed, of using the first letters of the alphabet for known and the last for unknown quantities. He also appears to have been the first to perceive that one general proof is sufficient for any proposition algebraically treated, the different cases which might arise by different arrangements of the equations being covered by the possibility of any letter representing a negative as well as a positive quantity, i.e., he distinguished the intrinsic sign of a quantity or symbol. Hitherto it had been considered necessary to treat separately the forms of the quadratic $ax^2 + bx = c$, $ax^2 = bx + c$, etc., which was a natural result of the geometric method of arriving at the solution. Descartes also introduced our present notation for powers, taking his exponents, however, only as positive and integral.

Contemporaneously with Descartes, Cavalieri, in Italy, applied the so-called "method of indivisibles" to the computation of areas, volumes, etc., a process which gave way early in the eighteenth century to the integral calculus At this time, also, the beginnings of the mathematical theory of probabilities were made by Pascal and Fermat in the solution of a certain problem which had been proposed.

A tremendous impulse was given to all branches of mathematics in the latter part of the seventeenth century by the genius of Newton. Besides his epoch-making discovery of the "theory of fluxions," or differential calculus, he contributed to algebraic science the idea of the general exponent or nth power (n being positive, negative, integral, or fractional), the binomial theorem, and a considerable part of the theory of equations.

To Leibnitz we owe the present notation of the differential calculus, the introduction of the terms "coördinates" and "axes of co-ordinates," and suggestions as to the use of indeterminate coefficients and determinants, which, though not developed by him, led, in the hands of others, to important results.

Jacob Bernoulli developed the fundamental principles of the calculus of probabilities, and made the first systematic attempts to construct an integral calculus. His brother John developed the exponential calculus, and treated trigonometry independently as a branch of analysis, it having been previously regarded as an adjunct of astronomy. The possibility of a calculus of operations was first recognized by Brook Taylor, after whom "Taylor's theorem" is named. De Moivre contributed to the discussion of imaginaries the important theorem which bears his name. In 1748 MacLaurin published an algebra which contained the results of some earlier papers published by him, among others one on the number of imaginary roots of an equation, and one on the determination of equal roots by means of the first derivative.

In the latter part of the eighteenth and beginning of the nineteenth centuries mathematical advancement was rapid under the powerful hands of Euler, Lagrange, Laplace, and Legendre. To these great men we owe the calculus of variations, the initial discussion of the calculus of imaginaries (which was afterwards systematized and developed by Gauss, Cauchy, and others), the treatment of determinants, contributions to the theory of equations, a large part of the integral calculus and differential equations, the development of the theory of probabilities, the treatment of elliptic functions, the method of least squares, and the specially algebraic treatment of the theory of numbers. In this list are included only those things which are of an algebraic nature.

We have now reached the beginning of our own century, in which the advance has been so rapid in all directions as to preclude more than a mere indication of some of the lines along which this has taken place, without any attempt at an enumeration of the illustrious names of those who have so magnificently carried forward the work.

The theory of equations has been perfected by the full use of the complex unit $a + bi$, forming thus, in the words of Cayley, a "universe complete in itself, such that, starting in it, we are never led out of it." We have, in fact, a double algebra as the instrument for the complete treatment of all higher analysis, except that in which one of higher multiplicity is used. The field of quantics has been brilliantly cultivated by Cayley, Sylvester, and others. The theory of matrices has been developed by Cayley, and it was shown by Professor J. Willard Gibbs, in his vice-presidential address before this section at the Buffalo meeting in 1886, that the simple and natural expression of this theory is in the language of multiple algebra. The φ of Hamilton is a matrix of the third order, and the Q of Grassmann a matrix of the nth order.

In the treatment of differential equations we have an algebra of operations, due primarily to George Boole, carried to a high degree of perfection, in which the symbol of differentiation is treated precisely as if it were a real quantity. In fact, we have come to regard scalar multiplication simply as a particular case in the calculus of operations which covers every possible case of the effect of one symbol upon another in producing some change in it. A further extension of this same idea we have in the algebra of logic, invented by the same author, and cultivated and extended by others since his time.

In conclusion, I propose to sketch briefly the development of the idea of a multiplicity of fundamental units, which is pervading more and more the mathematical thought of the day. This proceeded along two distinct lines, one arising from the interpretation of the imaginary, $\sqrt{-1}$, and the other entirely independent of this symbol or operation.

The first attempt to give a geometric meaning to the expression $a + bi$ appears to be due to Wallis in 1685, who proposed to construct the imaginary roots of a quadratic by going out of the line on which they would have been laid off if real. In 1804 the Abbé Buée devised the now accepted representation by laying off the terms containing i as a factor, at right angles to the others, and showed how to add and subtract such expressions as $a + bi$. At about the same time Argand published independently the same idea, and still further developed it. The concept of a directed quantity as represented by an algebraic symbol was thus necessarily arrived at. Gauss, Cauchy, and others have elaborated the complex unit more especially in the theory of numbers, while Euler, Peacock, De Morgan, and others have developed it more as a double algebra.

Up to this point i had been regarded as a scalar operator merely, and the corresponding geometry only plane, though attempts had been made without much success to extend the treatment into three-dimensional space. It remained for Hamilton to accomplish this by the simple device of making i a directed operator, or handle, perpendicular to the plane of rotation, which opened the way for any number of similar operators differing in direction, but, as to their other properties, simply square roots of minus one. In order to produce a convenient algebra on this basis, Hamilton was obliged to take the further step of giving to all vectors the properties of $\sqrt{-1}$, and thus the calculus of quaternions was produced, a non-commutative quadruple algebra. These ideas have been generalized still farther by Unverzagt in his "Theorie der goniometrischen und der longimetrischen Quaternionen." In this book the author first develops a trigonometry based on a general instead of a right-angled triangle, and then shows that the operator $j = (-1)^{\frac{\lambda}{\pi}}$ (in which λ is the fundamental angle, taking the place of $\frac{\pi}{2}$) takes in this trigonometry the place of i in De Moivre's theorem generalized. He then takes three units j_1, j_2, j_3, corresponding to Hamilton's i, j, k, and forms a generalized quaternion, based on some angle λ, which reduces to the ordinary system when $\lambda = \frac{\pi}{2}$. The case particularly discussed is that in which $\lambda = 0$.

The theory and laws of linear, associative algebras, which includes quaternions as a particular case, have been thoroughly treated by Peirce in his work bearing that title.

We turn now to the other line along which multiple algebras have been developed. In 1827 Möbius published his "Barycentrische Calcül," in which points are the ultimate units, to which any desired weights may be assigned. He gave the laws of combination of these units so far as addition and subtraction are concerned, but did not proceed to multiplication: in fact, he distinctly states that they can be multiplied only by numbers. He then proceeds to treat analytical geometry on this basis. His treatment of points, so far as it goes, is on the same plan afterwards independently developed by Grassmann.

In 1844, one year after Hamilton's first announcement of his discovery, Grassmann published his "Ausdehnungslehre," which contains a complete and logical exposition of his new algebra for any number of independent units, and hence, geometrically interpreted, for space of any dimensions. This book was so abstract and general in form, and so unlike the ordinary language of mathematics, that it attracted hardly any notice, and the author was obliged to recast and republish it in 1862. Grassman's algebra is non-linear, and only partially associative, so that it differs fundamentally from all those discussed by Peirce. The $\sqrt{-1}$ plays no part whatever in the theory, and Grassman's vector is a vector pure and simple, i.e., a quantity having direction and magnitude, and not, as in quaternions, a *versor*-vector, combining the properties of a vector and of the $\sqrt{-1}$. The fundamental notion of Grassmann's multiplication is extension or generation; the product $p_1 p_2$ is the line generated by a point moving straight from p_1 to p_2, etc.

In this great invention of Grassman we have a multiple algebra which is the natural language of geometry and mechanics, dealing in a manner astonishingly simple, concise, and expressive with these subjects, and certain, it appears to me, to gain constantly in the appreciation of mathematicians as it is more generally understood and used. The fact of its perfect adaptability to n-dimensional space is an additional argument in its favor for those who are interested in that line of investigation.

We have now traced the development of our subject from its elementary beginnings through a long period in which it was in the rhetorical stage, approaching at intervals here and there to the syncopated; then, on the revival of learning in Europe after the dark ages, we have seen its comparatively rapid progress through the syncopated stage to the purely symbolical, when it was at last in a shape suitable for the astonishing progress of the last two hundred years. Finally, in the present century, we have noted the appearance, as in the fulness of time, of multiple algebras from different and independent sources, whose realm is that of the future.

NOTES AND NEWS.

THE astronomers sent to the Sandwich Islands recently on the part of the International Geodetic Association of Europe and the United States Coast and Geodetic Survey, in order to make a more exhaustive study of the changes of latitude, have located their observatories at Waikiki, near Honolulu. It is proposed to observe during the year about sixty-five pairs of stars, chosen on account of their well-determined proper motions, and to make in all not far from twenty-five hundred observations of the latitude. The results, compared with those made simultaneously in Europe and America, will settle definitely the question whether there is a real motion of the pole. At the suggestion of the American representative, the force of gravity will be measured every night that latitude observations are made. This may throw light on one of the theories proposed to explain the changes of latitude, viz., that of large transfers of matter beneath the earth's surface. The new pendulums made at the Coast and Geodetic Survey Office in Washington, and which are similar to those taken to Alaska by Professor Mendenhall last spring, will be employed at Waikiki. They are of fine workmanship, and are capable of detecting changes that do not exceed one hundred-thousandth part of the quantity measured. Besides the observations at the regular station, a number of magnetic determinations will be made at other points in the Islands, — notably at Kealakeakua Bay, where Captain Cook observed the declination more than a hundred years ago, and at Lahaina, where De Freycinet had an observatory for pendulum and magnetic work in 1819. The re-occupation of these points will show the change of the needle during the past century, and will be of great value in determining the secular variation. It is intended also to seize the opportunity now presented to measure the force of gravity on the summit of Mauna Kea (14 000 feet elevation). Observations made at the top of Haleakala (10,000 feet) in 1887 showed conclusively that the mountain was solid. This fact received additional support from the zenith observations at the sea-level north and south of the mountain. The large deviation of the plumb line (29″) brought to light in that work has now been exceeded on Hawaii, where 1′ 26″ has been discovered at the south point of the island (Ka Lae). This fact, recently communicated by Surveyor-General Alexander, makes the question of the force of gravity at the summit of Mauna Kea one of double interest, and it is desirable, both from a geological and geodetic standpoint, that pendulum observations be made on top of one of the mountains. Doctor Marcuse, who is from the Royal Observatory at Berlin, observes for latitude on the part of the European association, and Mr. Preston, who made the observations at the summit of Haleakala four years ago, is from the United States Coast and Geodetic Survey, and makes gravity and magnetic determinations. He also, as the representative of the United States, observes for latitude in connection with Dr. Marcuse, in the international geodetic work. The observers had the good fortune to arrive at Honolulu on the day preceding the transit of Mercury (9th of May), and made successful observations of the phenomenon. The second contact was also observed by Mr. Lyons of the government survey. The two interior contacts were noted by local mean time (Waikiki 8″ east of Honolulu) as follows: —

	H.	M.	S.	H.	M.	S.
Mr. Lyons	1	26	32	—	—	—
Mr. Preston		26	53	6	10	50
Dr. Marcuse		27	3		11	22

The station was in latitude 21° 16′ 21″ north, and in longitude 157° 49′ 30″ west. The mean observed times of contact are in both cases about a minute less than the computed ones.