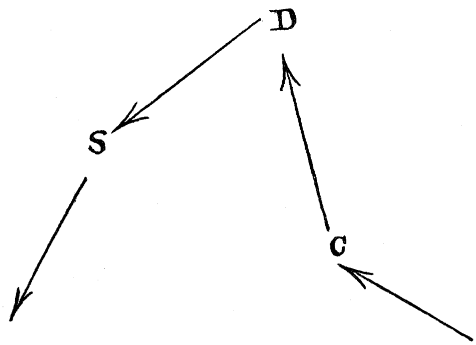
tion, as indicated by the arrows, would set in from *D* to *S*, one from *C* to *D*, one from *S* to some possibly South American Cambrian locality, and one, bringing a Permian or some later-day fauna, from an unknown locality towards *C*. Were this order of migration to continue here, or at other portions of the earth's surface, in this or in a similarly consecutive manner, the results obtained would be in perfect consonance with the facts presented by geology. But is there any reason whatever for the continuance of this order of migration? Surely no facts that have as yet been brought to light argue in favor of a continued migration in one direction. Why, then, it might justly be asked, could not just as well a migration take place from *S* to *D*, and impose with it a Silurian fauna upon a Devonian? What would there be to hinder



a migration from *S* to *C*, placing the American Silurian fauna upon the carboniferous of Africa? Why, as I have asked, has it just so happened that a fauna characteristic of a given period has *invariably* succeeded one which, when the two are in superposition all over the world (as far as we are aware), indicates precedence in creation or origination, and *never* one that can be shown to be of a later birth? Surely these peculiar circumstances cannot be accounted for on the doctrine of a fortuitous migration. And it certainly cannot be claimed that through a process of transmutation or development, depending upon the evolutionary forces, a fauna with a Silurian facies will, in the course of a possible migration toward a carboniferous locality, have assumed a carboniferous or Permian character.

The facts of geology and paleontology are, it appears to me, decidedly antagonistic to any such broad contemporaneity or non-contemporaneity as has been assumed by Professor Huxley; and their careful consideration will probably cause geologists to demur to the statement that "all competent authorities will probably assent to the proposition that physical geology does not enable us in any way to reply to this question: Were the British cretaceous rocks deposited at the same time as those of India, or are they a million of years younger or a million of years older?"

ANGELO HEILPRIN.

Academy of natural sciences,
Philadelphia, Dec. 8.

THOMSON AND TAIT'S NATURAL PHILOSOPHY.[1] — II.

BEFORE proceeding to an account of the rest of the work, we shall add a few more words of explanation upon the harmonic solutions of the differential equation (6), expressed in polar co-ordinates. On attempting to integrate this equation, it is found that there is an infinite number of particular solutions, as was before stated must necessarily be the fact; and each of these solutions is the product of three factors. One factor is an arbitrary constant; another factor is the radius vector raised to any integral power, positive or negative; and the remaining factor is a function of the angular co-ordinates, dependent for its form upon the exponent of that power of the radius vector by which it is multiplied. It is this last factor, or coefficient, which gives the name of 'spherical harmonics' to the solution: indeed, these functions of the angular co-ordinates are themselves surface-harmonics.

If we restrict ourselves, as is usually done, to real integral powers of the radius vector r, positive and negative, then, from the well-known principle that a general solution is obtained by taking the sum of particular solutions, we should have the most general possible solution by taking the sum of a series of particular solutions, such as have just been described, in which the powers of r have all integral values between $+\infty$ and $-\infty$. But since it is found, upon computing the functions of the angular co-ordinates which constitute their coefficients, that the coefficients of r^i and $r^{-(i+1)}$ are identical, it will be more convenient to write the general solution in the form —

$$V = a_0 f_0(\theta, \phi) + (a_1 r + b_1 r^{-2}) f_1(\theta, \phi) + (a_2 r^2 + b_2 r^{-3}) f_2(\theta, \phi) + \dots + (a_i r^i + b_i r^{-(i+1)}) f_i(\theta, \phi) + \dots \quad (8)$$

In applying this to any given case, either all the arbitrary constants a vanish, or all the constants b; thus giving rise to the two general forms of solution before mentioned, in which there is a series of terms, either in ascending integral powers of r, or of descending integral powers of r.

A value of V consisting of several terms is a compound spherical harmonic of the degree (positive or negative) of its numerically highest power of r. A value of V consisting of a single term is a simple harmonic.

Returning, now, to the consideration of chapter vii. p. 98, entitled 'Statics of solids and fluids,' the subject of rigid solids is disposed of in the course of thirty pages, nearly half of which is occupied with inextensible strings in the form of catenaries of various kinds.

The authors hasten on to the more intricate matter of elastic solids. As is well known to students of this subject, the general problem

[1] Concluded from No. 36.

of finding the displacements in all parts of an elastic solid of any figure subjected to the action of known forces applied to its exterior surfaces, even when the solid is uniform in texture in all directions (i.e., isotropic), transcends at present the powers of analysis, though considerable progress has been made toward a 'complete theory. An important contribution to this theory by Sir William Thomson is found on pp. 461 to 468 in Appendix C, entitled 'Equations of equilibrium of an elastic solid deduced from the principle, energy.'

By reason of the incompleteness of the general theory, those simple cases are first treated which are most completely amenable to analysis. The forty pages succeeding p. 130 treat the special case of the elastic wire, whose fundamental equations were first thoroughly investigated by Kirchhoff in 1859. This treatment, which is of interest both to the mathematician and engineer, investigates not only the spirals which elastic wires of circular and of rectangular cross-section assume under the action of direct forces, and of couples producing bending and twisting, but also goes into several important side-issues, one of which is the so-called kinetic analogy. A simple case of this, which is discussed at length, exists between the plane curves assumed by a thin flat spring, and the vibrations of a simple pendulum which it graphically represents. Another important side-issue is found in the discussion of the common spiral spring, in which the force resisting elongation is mostly due to torsion of the wire. Very curiously, the theorem of three moments of a straight beam is omitted, although the principles to be employed in establishing it are fully given.

Another important elastic solid which is fully amenable to analysis is the thin elastic plate. The treatment of the thin plate, which occupies thirty pages, discusses the flexure of a plane plate under all combinations of forces tending to produce either a state of synclastic stress (i.e., a state in which the curvature at every point is convex) or a state of anti-clastic stress (i.e., one which tends to cause the surface to become saddle-shaped). Kirchhoff's boundary conditions for a plate are also demonstrated at length. These are of importance in most practical cases, — as, for example, that of the flat steam-boiler head; for evidently any plate must have some kind of support or fastening at its boundary.

The general subject of elastic solids is reached at p. 204, and occupies a hundred pages, in which, after the general equations of equilibrium between the applied stresses and the resulting strains are established, several special cases are treated at length. The first of these is the celebrated torsion problem published by St. Venant in 1855; in which the distribution of the stresses and strains throughout a right prism of any cross-section whatever, under the action of forces applied to its ends, is completely determined. This is perhaps the most complicated problem which has been entirely worked out in the subject of elastic solids, and twenty-four pages are devoted to it. The flexure of beams having rectangular cross-sections is discussed, especially with reference to the distortions which are suffered by these cross-sections. The distortions can be easily exhibited by bending a thick rectangular piece of rubber, when the upper and lower surfaces will become saddle-shaped.

The general problem is then further treated by investigating the case of an infinite elastic solid under various suppositions as to the force applied through limited and through unlimited portions of it. The spherical and cylindrical shells are then treated by the help of harmonic analysis.

The concluding hundred and sixty pages of the work, beginning at p. 300, are devoted ostensibly to hydrostatics; but the first twenty-five pages finish those parts of the subject included under that title in ordinary treatises, and the remainder relates to the physics of the earth as dependent upon its fluid condition, past or present. The first great problem in this department of inquiry is to determine what figure will be assumed by a rotating liquid mass under the influence of centrifugal force and of the mutual gravitation of its parts. That an oblate spheroid is a figure of equilibrium for such a mass is commonly known, having been shown to be such by Newton; but that an ellipsoid with three unequal axes is also such a figure is not so commonly known, though this was discovered to be the fact by Jacobi in 1834. There are other possible figures, stable and unstable; but which of all these is the one which will actually be assumed in any given case? In reply to this question, the authors state, that "during the fifteen years which have passed since the publication of the first edition we have never abandoned the problem of the equilibrium of a finite mass of rotating incompressible fluid. Year after year, questions of the multiplicity of possible figures of equilibrium have been almost incessantly before us; and yet it is only now, under the compulsion of finishing this second edition of the second part of our first volume, with the hope for a second volume abandoned, that we have succeeded in finding any thing approaching full light on the subject" (p. 332). Then follows an enumeration of the possible forms of equilibrium, including the single and multiple rings into which an ellipsoid would be changed when rapidly rotated, and the detached portions, nearly spherical, into which an elongated ellipsoid must separate when rapidly rotated about its shorter diameter.

Now, on the supposition that the figure of the earth is approximately an oblate spheroid, the next matter of importance is to show how to compute the alterations in figure due to local inequalities in its density, and irregularities in the distribution of the material composing it. This at once raises the question as to what we are to consider as the surface of the earth at any point which forms part of its figure. The true figure of the earth may be taken to be the water-surface when undisturbed by tides. Whenever it is desired to find such surface on land, a canal could be supposed to be cut from the ocean to the place under consideration. Of course, a plumb-line is everywhere perpendicular to such a surface, whose outline is evidently affected by all existing inequalities of density and distribution of the substance of the earth. For example: it is computed that a set of several broad parallel mountain chains and valleys, which are twenty miles from crest to crest, and seventy-two hundred feet above the bottom of the valleys, would cause a corresponding undulation of the water-surface whose crests would be five feet above the bottoms of the hollows. This statement is equivalent to saying, that the plumb-line is deviated from its mean direction by the attraction of the mountain chains. Deviations of nearly 30″ have been actually observed near the Alps and near the Caucasus Mountains. The comparatively small deflections observed near the vast mass of the Himalayas in India — which, according to Pratt's calculations in his treatise on attractions, etc., should be vastly greater than any thing actually observed — indicate that extensive portions of the globe under those mountains are less than the average density. Localities have been found in flat countries also, notably in England and Russia, where the deflection of the plumb-line exceeds 15″, which is, of course, due to underlying material of great density. From this it appears, that the true figure of the earth is nearly as diversified as the contours of its hills and valleys, and does not correspond to any known geometrical figure; although, to be sure, these undulations are of small amount. Now, as a first rude approximation, the figure of the earth can be taken as a sphere, having the same volume as the actual earth. The earth at the equatorial regions will then project beyond the figure, and at the poles lie within it.

A second and better approximation can be made by taking the figure to be that of an oblate spheroid; and this is the basis upon which our present geodetic and astronomical measurements are based. Of course, it is possible to find an ellipsoid having three unequal axes which will coincide still more nearly with the results of observations upon the true figure of the earth; and this will furnish a third still closer approximation. This is what has been done by Capt. Clarke in his various publications. A summary of his results is given upon pp. 367 and 368.

It is evident, when the astronomical latitude is determined at any point of the earth's surface by measuring the elevation of the north pole above the horizon, as given by the spirit-level, that that determination will be in error by the entire amount of the local deviation of the plumb-line, which error may be as much as 30″, or more than half a mile, although the observations are made with all possible precision; and the outcome of geodetic triangulation may show that any such station whose position was supposed to have been determined astronomically to single feet really occupies a position, when referred to the spheroid, which at present furnishes the basis of all our astronomical and geodetic work, which is a considerable fraction of a mile from its position as so determined.

The last grand subject treated in the work is that of the tides on the corrected equilibrium theory, and matters closely connected with it. To explain what is meant by this, we shall briefly sketch the rise and progress of the theory of the tides.

Sir Isaac Newton, whose Principia appeared in 1687, showed that universal gravitation would not only account for the motions of the heavenly bodies in their orbits, but would also account for the tides, — phenomena whose cause had not, before his day, been traced to any simple law of nature. He showed that there would be a tide due to the attraction of the sun, and another to that of the moon, the latter being in general the larger; and that the actual tide would depend upon the relative position of those bodies, so that the highest or spring tides would be due to their combined effect, and the lowest or neap tides would occur when the tide due to the sun partially neutralized that of the moon. He showed how other known variations in the tide could be accounted for by the declinations of the sun and moon, and their greater or less distance from the earth.

The cause of the tide may be roughly stated, according to the equilibrium theory, thus: the sun or the moon attracts the water on the side of the earth nearer to it more than it does the earth itself, and attracts the earth itself more than the water on the farther side; the consequence being that water is heaped up on the sides of earth away from and toward the attracting body. Or, more exactly, we may imagine

"The rise and fall of the water at any point of the earth's surface to be produced by making two disturbing bodies (moon and anti-moon, as we may call them for brevity) revolve around the earth's axis once in the lunar twenty-four hours, with the line joining them always inclined to the earth's equator at an angle equal to the moon's declination. If we assume that at each moment the condition of hydrostatic equilibrium is fulfilled, — that is, that the free liquid surface is perpendicular to the rèsultant force, — we have what is called 'the equilibrium theory of the tides'" (art. 805).

Newton made a modification of this theory, which was intended to take into account the rotation of the earth, by supposing that the full effect of the attraction was not exerted immediately under the attracting body, but that the tide was of the nature of a wave, and by its inertia lagged behind the place where it should have been found in case the earth was not rotating. This retardation he thought might be more than a whole day in some cases. He was not able to submit the whole theory to rigorous computation for lack of sufficient data as to the mass of the moon and the height of the tides; but, from the tidal observations then available, he computed the mass of the moon necessary to produce them according to his theory, and obtained a result which we know to-day to be about twice too large.

In 1738 the French academy proposed the problem of the tides as the subject of a prize-essay, and elicited important essays on the subject from Bernouilli, Maclaurin, and Euler, to each of which was awarded a prize, and in each something of importance was added to Newton's theory; but the foundations of an exact and complete theory were first made in the 'Mécanique céleste' by Laplace, in five volumes, 1799–1825.

The science of mathematical analysis had not been greatly developed at the time Newton wrought upon this subject. His work is expressed in geometrical forms in which his genius is unapproachable. But the new methods of analysis founded upon the calculus, the principles of which were discovered equally by Newton and by Leibnitz, received a rapid and wonderful development during the seventeenth century at the hands of Lagrange and the continental mathematicians. It was to the then existing state of advancement in this particular that the great success of Laplace was due, which enabled him to unravel to so remarkable a degree the intricate interactions of the bodies of the solar system, and give for the first time the fundamental equations of the tides on correct principles. But it must be admitted that Laplace, in integrating his differential equations, seems to have become involved in intricate formulae whose full significance he has not correctly interpreted.

At about the same time, Dr. Thomas Young made an important investigation of the action of the tides, which was published in the Encyclopaedia Britannica, where it has been republished in succeeding editions to the present day. The special point of importance in his investigation was the discussion of the effect of friction upon the tides, which he showed to be such as to explain many difficulties, and that its magnitude might be such as to completely change the character of the tide at certain places so as to make low water take the place of high water, and *vice versa*, — a result hitherto unsuspected, and of prime importance.

The next great step in the theory of the tides was due to Airy, in his article on 'Tides and waves' in the Encyclopaedia metropolitana. He gave in new and concise form a most useful *résumé* of Laplace's theory, and made an original investigation of the effects of friction. He also made valuable additions to the theory as applied to shallow seas and rivers, a subject hitherto untouched.

The labors of Lubbock and of Dr. Whewell have added much to our knowledge of the relations of the theory to the observed tides; but the two foremost cultivators of this branch of science now living are Thomson and Ferrel. The former, who is chairman of the committee appointed by the British association for the advancement of science, for the purpose of the extension, improvement, and harmonic analysis of tidal observations, has done much, by his improved methods of observing tides and discussing them, to separate their components from each other, and render the exact comparison of theory and observed facts possible. Laplace assumed that the fortnightly and semi-annual tides due to the movement of the moon and sun in declination move so slowly that the equilibrium theory applies to them with exactness. But even if that be admitted, it can be shown that the theory needs correction to take account of the relative amount of land and water, as well as the contour of the continents. These have a controlling influence upon the tides, and this discovery is Thomson's great improvement and correction of the equilibrium theory.

The diurnal tide has been usually explained, in accordance with the equilibrium theory, as a wave existing under nearly static conditions, and following the moon and sun around the earth, but interfered with by friction, and changed in direction by the contour of the land. Though this was the view of Newton, Young, and others, and is incorporated in our ordinary text-books, it is quite inadequate; and the kinetic theory of Laplace must be put in its place, which treats the water as a moving fluid body, subject to the disturbing influence not only of the sun and moon, but of itself also.

The kinetic theory of the tides was to have been developed at length in vol. ii.; and that intended development is more than once referred to by the authors, — as, for instance, on p. 382, where an incidental comparison is made of the results of the two theories.

This part of the theory has been treated by Ferrel in his 'Tidal researches,' published as one of the appendices to the U. S. coast-survey report for 1874, in which work he has put in practical shape all the theoretical work heretofore accomplished, and also deduced therefrom important consequences. Until the publication of this work, it was not possible to apply the correct theory to the discussion and prediction of tides by reason of the unmanageable formulae employed by Laplace; and the discussions were, perforce, made by some modification of the equilibrium theory. Indeed, Laplace himself resorted to that method in his famous discussion of the tidal observations in the harbor of Brest. But, thanks to Ferrel's labors, this most intricate branch of computation has been systematized, and applied to an extensive series of tidal observations in Boston harbor.

The concluding pages, from 422 to 460, treat the question of the rigidity and solidity of the earth as a whole, especially as related to the tides. The final sentence (p. 460) is, "On the whole, we may fairly conclude, that, whilst there is some evidence of a tidal yielding of the earth's mass, that yielding is certainly small, and that the effective rigidity is at least as great as that of steel."

Four important papers on subjects related to those just mentioned are added to the work as appendices. The titles of these papers are, 'Cooling of the earth,' 'Age of the sun's heat,' 'Size of atoms,' 'Tidal friction.' The last three of these were not in the first edition.

WEEKLY SUMMARY OF THE PROGRESS OF SCIENCE.

MATHEMATICS.

Fuchsian functions. — A previous paper by M. Poincaré on this subject has already been noticed in these pages (i. **535**). In the present most important memoir, M. Poincaré assumes the results arrived at in the former memoir, and proceeds to more fully develop them and the consequences flowing from them. In the previous paper the author showed that it was possible to form discontinuous groups by substitutions of the form

$$\left(z, \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}\right)$$

by choosing the coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$, in such a way that the different substitutions of the group should not alter throughout the interior of a certain circle called the fundamental circle. In the present paper the author assumes that the fundamental circle has its centre at the origin, and its radius unity; so that its equation can be written as mod. $z = 1$.

He then considers one of these discontinuous groups, which he calls Fuchsian groups, and which he denotes by G. To this group corresponds a decomposition of the fundamental circle into an infinite number of normal polygons, R, all congruent among themselves. The author then demonstrates that there always exists a system of uniform functions of z, which remain unaltered by the different substitutions of the group G, and which he calls Fuchsian functions. M. Poincaré's memoir is too long to be reviewed here as it deserves. It is certainly a most important addition to the modern theory of functions, and is rendered particularly valuable by the historical note at the end, in which the author gives a brief account of the labors of Hermite, Fuchs, Klein, Schwarz, and others in this field. The two memoirs, with very little amplification, would constitute a really valuable treatise on this subject, — a subject of great importance, and on which there exists absolutely no text-book or treatise of any kind. — (*Acta math.*, i.) T. C. [**506**

ENGINEERING.

Steam-whistles. —Lloyd and Symes give a statement of experiments with a locomotive whistle having a bell $4\frac{15}{16}$ inches diameter, $3\frac{3}{4}$ inches long inside, and over an annular steam opening $\frac{1}{16}$ of an inch wide. The bell was of cast brass of medium character; and the lip was chamfered to a thin edge, and set exactly over the steam-opening. Sixty pounds press-