been inserted in this edition, embracing modern investigations of importance on this subject.

(*To be continued.*)

OBLIGATIONS OF MATHEMATICS TO PHILOSOPHY, AND TO QUESTIONS OF COMMON LIFE.[1] — II.

I SAID that I would speak to you, not of the utility of the mathematics in any of the questions of common life or of physical science, but rather of the obligations of mathematics to these different subjects. The consideration which thus presents itself is, in a great measure, that of the history of the development of the different branches of mathematical science in connection with the older physical sciences, — astronomy and mechanics. The mathematical theory is, in the first instance, suggested by some question of common life or of physical science, is pursued and studied quite independently thereof, and perhaps, after a long interval, comes in contact with it, or with quite a different question. Geometry and algebra must, I think, be considered as each of them originating in connection with objects or questions of common life, — geometry, notwithstanding its name, hardly in the measurement of land, but rather from the contemplation of such forms as the straight line, the circle, the ball, the top (or sugar-loaf). The Greek geometers appropriated for the geometrical forms corresponding to the last two of these the words *σφαῖρα* and *κῶνος*, our sphere and cone; and they extended the word 'cone' to mean the complete figure obtained by producing the straight lines of the surface both ways indefinitely. And so algebra would seem to have arisen from the sort of easy puzzles in regard to numbers which may be made, either in the picturesque forms of the Bija-Ganita, with its maiden with the beautiful locks, and its swarms of bees amid the fragant blossoms, and the one queen-bee left humming around the lotus-flower; or in the more prosaic form in which a student has presented to him in a modern text-book a problem leading to a simple equation.

The Greek geometry may be regarded as beginning with Plato (B.C. 430–347). The notions of geometrical analysis, loci, and the conic sections, are attributed to him; and there are in his 'Dialogues' many very interesting allusions to mathematical questions, — in particular the passage in the 'Theaetetus' where he affirms the incommensurability of the sides of certain squares. But the earliest extant writings are those of Euclid (B.C. 285). There is hardly any thing in mathematics more beautiful than his wondrous fifth book; and he has also, in the seventh, eighth, ninth, and tenth books, fully and ably developed the first principles of the theory of numbers, including the theory of incommensurables. We have next Apollonius (about B.C. 247) and Archimedes (B.C. 287–212), both geometers of the highest merit, and the latter of them the founder of the science of statics (including therein hydrostatics). His dictum about the lever, his '*Εὕρηκα*,' and the story of the defence of Syracuse, are well known. Following these we have a worthy series of names, including the astronomers Hipparchus (B.C. 150) and Ptolemy (A.D. 125), and ending, say, with Pappus (A.D. 400), but continued by their Arabian commentators, and the Italian and other European geometers of the sixteenth century and later, who pursued the Greek geometry.

The Greek arithmetic was, from the want of a proper notation, singularly cumbrous and difficult; and it was, for astronomical purposes, superseded by the sexagesimal arithmetic, attributed to Ptolemy, but probably known before his time. The use of the present so-called Arabic figures became general among Arabian writers on arithmetic and astronomy about the middle of the tenth century, but it was not introduced into Europe until about two centuries later. Algebra, among the Greeks, is represented almost exclusively by the treatise of Diophantus (A.D. 150), — in fact, a work on the theory of numbers, containing questions relating to square and cube numbers, and other properties of numbers, with their solutions. This has no historical connection with the later algebra introduced into Italy from the east by Leonardi Bonacci of Pisa (A.D. 1202–1208), and successfully cultivated in the fifteenth and sixteenth centuries by Lucas Paciolus, or de Burgo, Tartaglia, Cardan, and Ferrari. Later on, we have Vieta (1540–1603), Harriot, already referred to, Wallis, and others.

Astronomy is, of course, intimately connected with geometry. The most simple facts of observation of the heavenly bodies can only be *stated* in geometrical language; for instance, that the stars describe circles about the Pole-star, or that the different positions of the sun among the fixed stars in the course of the year form a circle. For astronomical calculations it was found necessary to determine the arc of a circle by means of its chord. The notion is as old as Hipparchus, a work of whom is referred to as consisting of twelve books on the chords of circular arcs. We have (A.D. 125) Ptolemy's 'Almagest,' the first book of which contains a table of arcs and chords, with the method of construction; and among other theorems on the subject, he gives there the theorem, afterwards inserted in Euclid (book vi. prop. D), relating to the rectangle contained by the diagonals of a quadrilateral inscribed in a circle. The Arabians made the improvement of using, in place of the chord of an arc, the sine, or half-chord of double the arc, and so brought the theory into the form in which it is used in modern trigonometry. The before-mentioned theorem of Ptolemy, — or, rather, a particular case of it, — translated into the notation of sines, gives the expression for the sine of the sum of two arcs in terms of the sines and cosines of the component arcs, and it is thus the fundamental theorem on the subject. We have in the fifteenth and sixteenth centuries a series of mathematicians, who, with wonderful enthusiasm and perseverance, calculated tables of the trigonometrical or circular functions, — Purbach, Müller or Regiomontanus,

[1] Address of Professor CAYLEY before the British association. Concluded from No. 35.

Copernicus, Reinhold, Maurolycus, Vieta, and many others. The tabulations of the functions tangent and secant are due to Reinhold and Maurolycus respectively.

Logarithms were invented, not exclusively with reference to the calculation of trigonometrical tables, but in order to facilitate numerical calculations generally. The invention is due to John Napier of Merchiston, who died in 1618, at sixty-seven years of age. The notion was based upon refined mathematical reasoning on the comparison of the spaces described by two points; the one moving with a uniform velocity, the other with a velocity varying according to a given law. It is to be observed that Napier's logarithms were nearly, but not exactly, those which are now called, sometimes Napierian, but more usually hyperbolic logarithms, those to the base e; and that the change to the base 10 (the great step by which the invention was perfected for the object in view) was indicated by Napier, but actually made by Henry Briggs, afterwards Savilian professor at Oxford (d. 1630). But it is the hyperbolic logarithm which is mathematically important. The direct function e^x, or exp. x, which has for its inverse the hyperbolic logarithm, presented itself, but not in a prominent way. Tables were calculated of the logarithms of numbers, and of those of the trigonometrical functions.

The circular function and the logarithm were thus invented each for a practical purpose, separately, and without any proper connection with each other. The functions are connected through the theory of imaginaries, and form together a group of the utmost importance throughout mathematics: but this is mathematical theory; the obligation of mathematics is for the discovery of the functions.

Forms of spirals presented themselves in Greek architecture, and the curves were considered mathematically by Archimedes. The Greek geometers invented some other curves more or less interesting, but recondite enough in their origin. A curve which might have presented itself to anybody, that described by a point in the circumference of a rolling carriage-wheel, was first noticed by Mersenne in 1615, and is the curve afterwards considered by Roberval, Pascal, and others, under the name of the roulette, otherwise the cycloid. Pascal (1623–62) wrote, at the age of seventeen, his 'Essais pour les coniques' in seven short pages, full of new views on these curves, and in which he gives, in a paragraph of eight lines, his theory of the inscribed hexagon.

Kepler (1571–1630), by his empirical determination of the laws of planetary motion, brought into connection with astronomy one of the forms of conic, the ellipse, and established a foundation for the theory of gravitation. Contemporary with him for most of his life, we have Galileo (1564–1642), the founder of the science of dynamics; and closely following upon Galileo, we have Isaac Newton (1643–1727). The 'Philosophiae naturalis principia mathematica,' known as the 'Principia,' was first published in 1687.

The physical, statical, or dynamical questions which presented themselves before the publication of the 'Principia' were of no particular mathematical difficulty; but it is quite otherwise with the crowd of interesting questions arising out of the theory of gravitation, and which, in becoming the subject of mathematical investigation, have contributed very much to the advance of mathematics. We have the problem of two bodies, or, what is the same thing, that of the motion of a particle about a fixed centre of force, for any law of force; we have also the problem (mathematically very interesting) of the motion of a body attracted to two or more fixed centres of force; then, next preceding that of the actual solar system, the problem of three bodies. This has ever been and is far beyond the power of mathematics; and it is in the lunar and planetary theories replaced by what is mathematically a different problem, — that of the motion of a body under the action of a principal central force and a disturbing force, — or, in one mode of treatment, by the problem of disturbed elliptic motion. I would remark that we have here an instance in which an astronomical fact, the observed slow variation of the orbit of a planet, has directly suggested a mathematical method, applied to other dynamical problems, and which is the basis of very extensive modern investigations in regard to systems of differential equations. Again: immediately arising out of the theory of gravitation, we have the problem of finding the attraction of a solid body of any given form upon a particle, solved by Newton in the case of a homogeneous sphere, but which is far more difficult in the next succeeding cases of the spheroid of revolution (very ably treated by Maclaurin), and of the ellipsoid of three unequal axes. There is, perhaps, no problem of mathematics which has been treated by so great a variety of methods, or has given rise to so much interesting investigation, as this last problem of the attraction of an ellipsoid upon an interior or exterior point. It was a dynamical problem, that of vibrating strings, by which Lagrange was led to the theory of the representation of a function as the sum of a series of multiple sines and cosines; and connected with this we have the expansions in terms of Legendre's functions P_n, suggested to him by the question, just referred to, of the attraction of an ellipsoid. The subsequent investigations of Laplace, on the attractions of bodies differing slightly from the sphere, led to the functions of two variables called Laplace's functions. I have been speaking of ellipsoids; but the general theory is that of attractions, which has become a very wide branch of modern mathematics. Associated with it, we have in particular the names of Gauss, Lejeune-Dirichlet, and Green; and I must not omit to mention that the theory is now one relating to n-dimensional space. Another great problem of celestial mechanics, that of the motion of the earth about its centre of gravity (in the most simple case, that of a body not acted upon by any forces), is a very interesting one in the mathematical point of view.

I may mention a few other instances where a practical or physical question has connected itself with the development of mathematical theory. I have spoken of two map projections, — the stereographic, dating from Ptolemy; and Mercator's projection, invented by Edward Wright about the year 1600. Each of these, as a particular case of the orthomorphic projection, belongs to the theory of the geometrical representation of an imaginary variable. I have spoken also of perspective, and (in an omitted paragraph) of the representation of solid figures employed in Monge's descriptive geometry. Monge, it is well known, is the author of the geometrical theory of the curvature of surfaces, and of curves of curvature. He was led to this theory by a problem of earthwork, — from a given area, covered with earth of uniform thickness, to carry the earth and distribute it over an equal given area with the least amount of cartage. For the solution of the corresponding problem in solid geometry, he had to consider the intersecting normals of a surface, and so arrived at the curves of curvature (see his 'Mémoire sur les déblais et les remblais,' *Mém. de l'acad.*, 1781). The normals of a surface are, again, a particular case of a doubly infinite system of lines, and are so connected with the modern theories of congruences and complexes.

The undulatory theory of light led to Fresnel's wave-surface, — a surface of the fourth order, by far the most interesting one which had then presented itself. A geometrical property of this surface, that of having tangent planes, each touching it along a plane curve (in fact, a circle), gave to Sir W. R. Hamilton the theory of conical refraction. The wave-surface is now regarded in geometry as a particular case of Kummer's quartic surface, with sixteen conical points and sixteen singular tangent planes.

My imperfect acquaintance, as well with the mathematics as the physics, prevents me from speaking of the benefits which the theory of partial differential equations has received from the hydrodynamical theory of vortex motion, and from the great physical theories of electricity, magnetism, and energy.

It is difficult to give an idea of the vast extent of modern mathematics. This word 'extent' is not the right one: I mean extent crowded with beautiful detail, — not an extent of mere uniformity, such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through, and studied in every detail of hillside and valley, stream, rock, wood, and flower. But as for any thing else, so for a mathematical theory, — beauty can be perceived, but not explained. As for mere extent, I might illustrate this by speaking of the dates at which some of the great extensions have been made in several branches of mathematical science.

And, in fact, in the address as written, I speak at considerable length of the extensions in geometry since the time of Descartes, and in other specified subjects since the commencement of the century. These subjects are the general theory of the function of an imaginary variable; the leading known functions, viz., the elliptic and single theta-functions and the Abelian and multiple theta-functions; the theory of equations and the theory of numbers. I refer also to some theories outside of ordinary mathematics, — the multiple algebra, or linear associative algebra, of the late Benjamin Peirce; the theory of Argand, Warren, and Peacock, in regard to imaginaries in plane geometry; Sir W. R. Hamilton's quaternions; Clifford's biquaternions; the theories developed in Grassmann's 'Ausdehnungslehre,' with recent extensions thereof to non-Euclidian space by Mr. Homersham Cox; also Boole's 'Mathematical logic,' and a work connected with logic, but primarily mathematical and of the highest importance, Shubert's 'Abzählende geometrie' (1878). I remark that all this in regard to theories outside of ordinary mathematics is still on the text of the vast extent of modern mathematics.

In conclusion, I would say that mathematics have steadily advanced from the time of the Greek geometers. Nothing is lost or wasted. The achievements of Euclid, Archimedes, and Apollonius, are as admirable now as they were in their own days. Descartes' method of co-ordinates is a possession forever. But mathematics has never been cultivated more zealously and diligently, or with greater success, than in this century, — in the last half of it, or at the present time. The advances made have been enormous. The actual field is boundless, the future full of hope. In regard to pure mathematics we may most confidently say, —

"Yet I doubt not through the ages one increasing purpose runs,
And the thoughts of men are widened with the process of the suns."

THE ENDOWMENT OF BIOLOGICAL RESEARCH.[1]

It has become the custom for the presidents of the various sections of this association to open the proceedings of the departments with the chairmanship of which they are charged by formal addresses. In reflecting on the topics which it might be desirable for me to bring under your notice, as your president, on the present occasion, it has occurred to me that I might use this opportunity most fitly by departing somewhat from the prevailing custom of reviewing the progress of science in some special direction during the past year, and that, instead of placing before you a summary of the results recently obtained by the investigations of biologists in this or that line of inquiry, I might ask your attention, and that of the external public (who are wont to give some kindly consideration to the opinions expressed on these occasions) to a matter which is even more directly connected with the avowed object of our association; namely, 'the advancement of science.' I propose to place before you a few observations upon the provision which exists in this country for the advancement of that branch of science to which section D is dedicated; namely, biology.

I am aware that it is usual for those who speak of men of science and their pursuits to ignore altogether such sordid topics as the one which I have chosen to bring forward. A certain pride, on the one hand, and a willing acquiescence, on the other hand, usually prevent those who are professionally concerned with

[1] An address to the biological section of the British association. By Prof. E. Ray Lankester, M.A., F.R.S., F.L.S., president of the section. From advance copy kindly furnished by the editor of *Nature*.